

PROJECTIVE MODEL STRUCTURES FOR EXACT CATEGORIES

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ABSTRACT. In this article we provide sufficient conditions on weakly idempotent complete exact categories \mathcal{E} which admit an abelian embedding, such that various categories of chain complexes in \mathcal{E} are equipped with projective model structures. In particular we show that as soon as \mathcal{E} has enough projectives, the category $\mathbf{Ch}_+(\mathcal{E})$ of bounded below complexes is equipped with a projective model structure. In the case that \mathcal{E} also admits all kernels we show that it is also true of $\mathbf{Ch}_{\geq 0}(\mathcal{E})$, and that a generalisation of the Dold-Kan correspondence holds. Supplementing the existence of kernels with a condition on the existence and exactness of certain direct limit functors guarantees that the category of unbounded chain complexes $\mathbf{Ch}(\mathcal{E})$ also admits a projective model structure. When \mathcal{E} is monoidal we also examine when these model structures are monoidal and conclude by studying some homotopical algebra in such categories. Along the way we also discuss generators in exact categories.

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1. INTRODUCTION

Let k be a field of characteristic 0. It is well known (see [Hov07]) that there is a projective model structure on the category $\mathbf{Ch}(k)$ of unbounded chain complexes in the abelian category \mathbf{Vect}_k of k -vector spaces in which weak equivalences are quasi-isomorphisms and fibrations are degreewise epimorphisms. $\mathbf{Ch}(k)$ is equipped with a natural monoidal structure, inherited from the one on \mathbf{Vect}_k . It is also known that the category \mathbf{cdga}_k of commutative differential graded algebras over k inherits a model structure from $\mathbf{Ch}(k)$ (this is shown, for example, in [Lur11]). This model category is equipped with various homotopy Grothendieck topologies arising from the monoidal structure. In arbitrary characteristic, one can also work in the category \mathbf{sVect}_k of simplicial k -vector spaces, see [Qui06].

Lately there has been much progress in developing theories of derived analytic geometry. In recent work of Ben-Bassat, Kremnizer and Bambozzi, namely [BBK13], [BBB15], [Bam14] and [BBBK15] the authors develop a functor of points approach to analytic geometry. They begin with a normed field k , and consider both the monoidal categories $\mathbf{Ind}(\mathbf{Ban}_k)$ of indizations of Banach spaces over k , and \mathbf{CBorn}_k of complete bornological spaces of convex type over k . These categories are not quite abelian. However they are quasi-abelian. Quasi-abelian categories are a sub-class of the class of so-called Quillen exact categories. These are additive categories equipped with a suitable notion of homology. In particular, in [BBBK15] the authors use this homological structure to construct a Grothendieck topology on a subcategory \mathbf{St} of $(\mathbf{Comm}(\mathbf{CBorn}_k))^{op}$. \mathbf{St} is equivalent to the category of (dagger) Stein spaces, and when $k = \mathbb{C}$ the coverings in their topology correspond to coverings of Stein spaces by Stein spaces. They mention also that there should be a model structure together with a suitable homotopy Grothendieck topology on the category of commutative simplicial algebras in \mathbf{CBorn}_k , $\mathbf{Comm}(\mathbf{sCBorn}_k)^{op}$ which induces the topology on \mathbf{St} . In particular, this gives one possible direction for developing derived analytic geometry.

Instead of simplicial objects, in this paper we shall consider model structures on chain complexes in quasi-abelian, and more generally exact categories \mathcal{E} . Our primary goal is to establish the existence of projective model structures under suitable conditions on categories of chain complexes in \mathcal{E} , and to study some homotopical algebra in such categories. The strategy is as follows. In [Hov02], Hovey developed a method for endowing abelian categories \mathcal{A} with model structures using the homological data of Hovey Triples. Gillespie noted in [Gil11] that this works for (weakly idempotent complete) exact categories as well. Hovey also discusses when Hovey Triples produce cofibrantly small model structures, and monoidal model structures in the case that \mathcal{A} is monoidal. We show that these results also generalise to exact categories. In [Gil04], [Gil07] and [Gil08] Gillespie develops a method for constructing a Hovey Triple on chain complexes in an abelian category \mathcal{A} from a cotorsion pair on \mathcal{A} . Again we generalise this to bounded below complexes in exact categories, and assuming some mild extra assumptions on \mathcal{E} , unbounded complexes as well.

This paper is organised as follows. In Section 2 we recall some basic facts about exact categories following [Büh10] and discuss various notions of acyclicity in such categories. We also give our definition of a monoidal exact category, Definition 2.67.

In Section 3 we discuss generators and introduce elementary exact categories. The main technical result of this section is Corollary 3.17, which says that categories of chain complexes in elementary exact categories are themselves elementary. We also study generators in categories of algebras for monads.

In Section 4 we recall Hovey's (or Gillespie's in the case of exact categories) bijection between Hovey Triples and compatible model structures. Next we generalise Hovey's results on cofibrant generation (Lemma 4.16) and monoidal model structures (Theorem 4.18, Lemma 4.22, Theorem 4.24) to the exact case. Finally we show that Gillespie's method for constructing Hovey Triples on chain complexes can also be generalised to the exact case (Subsection 4.5).

In Section 5 we apply the results of Section 4 to construct projective model structures. Assuming only the existence of enough projectives we obtain such a model structure on $\mathbf{Ch}_+(\mathcal{E})$ (Theorem 5.3). If in addition we allow \mathcal{E} to have kernels, then we get a model structure on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ (Theorem 5.7). We show that when the exact category is elementary, these model structures are cofibrantly small. For monoidal elementary exact categories we show under some mild extra assumptions that the resulting model structures on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ and $\mathbf{Ch}(\mathcal{E})$ are monoidal and satisfy the monoid axiom (Theorem 5.13 and Theorem 5.14).

Finally in section 6 we discuss categories of monoids in monoidal exact categories equipped with compatible model structures. More generally we discuss model structures on monoidal additive categories which make them suitable settings for homotopical algebra. In particular, we show that for such a category, the categories of commutative monoids and Lie monoids are equipped with transferred model structures (Proposition 6.4 and Proposition 6.5).

In Appendix A we review some general facts concerning algebra in symmetric monoidal additive categories, while Appendix B contains important results about model categories which we use throughout this work.

1.1. Notation and Conventions.

1.1.1. *Categories.* Throughout this work we will use the following notation.

- \mathbf{Ab} is the abelian category of abelian groups.
- If k is a field, then \mathbf{Vect}_k denotes the category of vector spaces over k .
- Unless stated otherwise, the unit in a monoidal category will be denoted by k , the tensor functor by \otimes , and for a closed monoidal category the internal hom functor will be denoted by $\underline{\mathrm{Hom}}$. Monoidal categories will always be assumed to be symmetric, with symmetric braiding σ .
- If \mathcal{C}, \mathcal{D} are categories then $[\mathcal{C}, \mathcal{D}]$ denotes the category of covariant functors between them.
- Filtered colimits will be denoted by \lim_{\rightarrow} .
- We will say a category is bicomplete if it has all limits and colimits.

1.1.2. *Chain Complexes.* Let us now introduce some conventions for chain complexes.

Definition 1.1. A *chain complex* in a preadditive category \mathcal{E} is a sequence

$$K_{\bullet} = \dots \longrightarrow K_n \xrightarrow{d_n} K_{n-1} \xrightarrow{d_{n-1}} K_{n-2} \longrightarrow \dots$$

where the K_i are objects and the d_i are morphisms such that $d_{n-1} \circ d_n = 0$. The morphisms are called *differentials*.

A *morphism of chain complexes* $f_{\bullet} : K_{\bullet} \rightarrow L_{\bullet}$ is a collection of morphisms $f_n : K_n \rightarrow L_n$ such that the following diagram commutes for each n :

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & K_{n+1} & \xrightarrow{d_{n+1}^K} & K_n & \xrightarrow{d_n^K} & K_{n-1} \longrightarrow \cdots \\
& & \downarrow f_{n+1} & & \downarrow f^n & & \downarrow f^{n-1} \\
\cdots & \longrightarrow & L_{n+1} & \xrightarrow{d_{n+1}^L} & L_n & \xrightarrow{d_n^L} & L_{n-1} \longrightarrow \cdots
\end{array}$$

The category whose objects are chain complexes and whose morphisms are as described above is called the **category of chain complexes in \mathcal{E}** , denoted $\mathbf{Ch}(\mathcal{E})$. We also define $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ to be the full subcategory of $\mathbf{Ch}(\mathcal{E})$ on complexes A_\bullet such that $A_n = 0$ for $n < 0$, $\mathbf{Ch}_{\leq 0}(\mathcal{E})$ to be the full subcategory of $\mathbf{Ch}(\mathcal{E})$ on complexes A_\bullet such that $A_n = 0$ for $n > 0$, $\mathbf{Ch}_+(\mathcal{E})$, the full subcategory of chain complexes A_\bullet such that $A_n = 0$ for $n < 0$, $\mathbf{Ch}_-(\mathcal{E})$, the full subcategory of chain complexes A_\bullet such that $A_n = 0$ for $n > 0$ and $\mathbf{Ch}_b(\mathcal{E})$ to be the full subcategory of $\mathbf{Ch}(\mathcal{E})$ on complexes A_\bullet such that $A_n \neq 0$ for only finitely many n . A lot of the statements in the rest of this document apply to several of these categories at once. In such cases we will write $\mathbf{Ch}_*(\mathcal{E})$, and specify that $*$ can be any element of some subset of $\{\geq 0, \leq 0, +, -, b, \emptyset\}$, where by definition $\mathbf{Ch}_{\emptyset}(\mathcal{E}) = \mathbf{Ch}(\mathcal{E})$.

All of the above categories are naturally enriched over $\mathbf{Ch}(\mathbf{Ab})$. We denote the enriched hom by $\mathbf{Hom}(-, -)$. For notational clarity we recall its definition here.

Definition 1.2. Let $X_\bullet, Y_\bullet \in \mathbf{Ch}(\mathcal{E})$. We define $\mathbf{Hom}(X_\bullet, Y_\bullet) \in \mathbf{Ch}(\mathbf{Ab})$ to be the complex with

$$\mathbf{Hom}(X_\bullet, Y_\bullet)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{E}}(X_i, Y_{i+n})$$

and differential d_n defined on $\mathrm{Hom}_{\mathcal{E}}(X_i, Y_{i+n})$ by

$$df = d_{i+n}^Y \circ f - (-1)^n f \circ d_i^X$$

We will also frequently use the following special chain complexes.

Definition 1.3. If E is an object of an exact category \mathcal{E} we let $S^n(E) \in \mathbf{Ch}(\mathcal{E})$ be the complex whose n th entry is E , with all other entries being 0. We also denote by $D^n(E) \in \mathbf{Ch}(\mathcal{E})$ the complex whose n th and $(n-1)$ st entries are E , with all other entries being 0, and the differential d_n being the identity.

Let $(\mathcal{E}, \otimes, k)$ be a monoidal additive category, i.e. \otimes is an additive bifunctor. There is an induced monoidal structure on $\mathbf{Ch}_*(\mathcal{E})$ for $*$ in $\{\geq 0, \leq 0, +, -, b, \emptyset\}$. The unit is $S^0(k)$. If X_\bullet and Y_\bullet are chain complexes then we set

$$(X_\bullet \otimes Y_\bullet)_n = \bigoplus_{i+j=n} X_i \otimes Y_j$$

If $i + j = n$, then we define the differential on the summand $X_i \otimes Y_j$ of $(X_\bullet \otimes Y_\bullet)_n$ by

$$d_n^{X_\bullet \otimes Y_\bullet}|_{X_i \otimes Y_j} = d_i^{X_\bullet} \otimes id_{Y_j} + (-1)^i id_{X_i} \otimes d_j^{Y_\bullet}$$

If $*$ in $\{\geq 0, \leq 0, +, -, b, \emptyset\}$ then $(\mathbf{Ch}_*(\mathcal{E}), \otimes, S^0(k))$ is a monoidal additive category.

If $(\mathcal{E}, \otimes, k, \underline{\mathbf{Hom}})$ is a closed monoidal additive category then we define a functor

$$\underline{\mathbf{Hom}}(-, -) : \mathbf{Ch}(\mathcal{E})^{op} \times \mathbf{Ch}(\mathcal{E}) \rightarrow \mathbf{Ch}(\mathcal{E})$$

$$\underline{\mathbf{Hom}}(X_\bullet, Y_\bullet)_n = \prod_{i \in \mathbb{Z}} \underline{\mathbf{Hom}}_{\mathcal{E}}(X_i, Y_{i+n})$$

and differential d_n defined on $\underline{\mathbf{Hom}}_{\mathcal{E}}(X_i, Y_{i+n})$ by

$$d = \underline{\mathbf{Hom}}(d_i^{X_\bullet}, id) + (-1)^i \underline{\mathbf{Hom}}(id, d_{i+n}^{Y_\bullet})$$

This does define an internal hom on the monoidal category

$$(\mathbf{Ch}(\mathcal{E}), \otimes, S^0(k))$$

The internal hom on chain complexes also restricts to a bifunctor

$$\underline{\mathbf{Hom}}(-, -) : \mathbf{Ch}_b(\mathcal{E})^{op} \times \mathbf{Ch}_b(\mathcal{E}) \rightarrow \mathbf{Ch}_b(\mathcal{E})$$

Then

$$(\mathbf{Ch}_b(\mathcal{E}), \otimes, S^0(k), \underline{\mathbf{Hom}})$$

is a closed monoidal additive category. In fact, in both of these categories there are natural isomorphisms of chain complexes of abelian groups.

$$\mathbf{Hom}(X_\bullet, \underline{\mathbf{Hom}}(Y_\bullet, Z_\bullet)) \cong \mathbf{Hom}(X_\bullet \otimes Y_\bullet, Z_\bullet)$$

The categories $\mathbf{Ch}_*(\mathcal{E})$ for $*$ in $\{+, -, b, \emptyset\}$ also come equipped with a shift functor. It is given on objects by

$$(A_\bullet[1])_i = A_{i+1}$$

with differential given by

$$d_i^{A[1]} = -d_{i+1}^A$$

The shift of a morphism f_\bullet is given by $(f_\bullet[1])_i = f_{i+1}$. $[1]$ is an auto-equivalence with inverse $[-1]$. We set $[0] = \text{Id}$ and $[n] = [1]^n$ for any integer n .

Finally, we define the mapping cone as follows.

Definition 1.4. Let X_\bullet and Y_\bullet be chain complexes in an additive category \mathcal{E} and $f_\bullet : X_\bullet \rightarrow Y_\bullet$. The **mapping cone of f_\bullet** , denoted $\text{cone}(f_\bullet)$ is the complex whose components are

$$\text{cone}(f_\bullet)_n = X_{n-1} \oplus Y_n$$

and whose differential is

$$d_n^{\text{cone}(f)} = \begin{pmatrix} -d_{n-1}^X & 0 \\ -f_{n-1} & d_n^Y \end{pmatrix}$$

There are natural morphisms $\tau : Y_\bullet \rightarrow \text{cone}(f)$ induced by the injections $Y_i \rightarrow X_{i-1} \oplus Y_i$, and $\pi : \text{cone}(f) \rightarrow X_\bullet[-1]$ induced by the projections $X_{i-1} \oplus Y_i \rightarrow X_{i-1}$. The sequence

$$Y_\bullet \rightarrow \text{cone}(f) \rightarrow X_\bullet[-1]$$

is split exact in each degree.

Finally let us introduce some notation for truncation functors.

Definition 1.5. Let \mathcal{E} be an additive category which has kernels. For a complex X_\bullet we denote by $\tau_{\geq n}X$ the complex such that $(\tau_{\geq n}X)_m = 0$ if $m < n$, $(\tau_{\geq n}X)_m = X_m$ if $m > n$ and $(\tau_{\geq n}X)_n = \text{Ker}(d_n)$. The differentials are the obvious ones. The construction is clearly functorial.

2. EXACT CATEGORY GENERALITIES

In this section we review the main notions of exact categories, following [Büh10]. In the following \mathcal{E} will be an additive category. A **kernel-cokernel pair** in \mathcal{E} is a pair of composable maps (i, p) , $i : A \rightarrow B$, $p : B \rightarrow C$ such that $i = \text{Ker}(p)$ and $p = \text{Coker}(i)$. If \mathcal{Q} is a class of kernel-cokernel pairs and $(i, p) \in \mathcal{Q}$, then we say that i is an admissible monic and p is an admissible epic with respect to \mathcal{Q} .

Definition 2.1. A **Quillen exact structure** on an additive category \mathcal{E} is a collection \mathcal{Q} of kernel-cokernel pairs such that

- (1) Isomorphisms are both admissible monics and admissible epics.
- (2) Both the collection of admissible monics and the collection of admissible epics are closed under composition.

(3) If

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f'} & Y \end{array}$$

is a push out diagram, and f is an admissible monic, then f' is as well.

(4) If

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback diagram, and f is an admissible epic, then f' is as well.

Let $(\mathcal{E}, \mathcal{Q})$ be an exact category. We call a null sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

short exact if (i, p) is a kernel-cokernel pair in \mathcal{Q} . We will use interchangeably the notion of kernel-cokernel pair and short exact sequence. In the context of diagrams in exact categories \rightarrow will be used to denote an admissible monic, and \twoheadrightarrow an admissible epic. When it is not likely to cause confusion, we will suppress the notation $(\mathcal{E}, \mathcal{Q})$ to \mathcal{E} .

When studying exact categories it is natural to consider so-called exact functors:

Definition 2.2. Let $(\mathcal{E}, \mathcal{P})$, $(\mathcal{F}, \mathcal{Q})$ be exact categories. A functor $F : \mathcal{E} \rightarrow \mathcal{F}$ is said to be **exact** (with respect to \mathcal{P} and \mathcal{Q}) if for any short exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

in \mathcal{P} ,

$$0 \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow 0$$

is a short exact sequence in \mathcal{Q} .

Definition 2.3. Let $(\mathcal{E}, \mathcal{P})$ be an exact category. An **exact subcategory** of $(\mathcal{E}, \mathcal{P})$ is an exact category $(\mathcal{F}, \mathcal{Q})$ where \mathcal{F} is a subcategory of \mathcal{E} and the inclusion functor is exact.

Note that in an exact category $(\mathcal{E}, \mathcal{Q})$ the map $A \rightarrow 0$ is an admissible epic and the map $0 \rightarrow A$ is an admissible monic for any object A . This follows from the fact that $id : A \rightarrow A$ is both an admissible epic and an admissible monic, and the fact that

$$0 \rightarrow 0 \rightarrow A \rightarrow A \rightarrow 0$$

and

$$0 \rightarrow A \rightarrow A \rightarrow 0 \rightarrow 0$$

are short exact sequences. As a consequence any split exact sequence

$$0 \rightarrow A \rightarrow A \oplus B \rightarrow B \rightarrow 0$$

is short exact. Indeed in the following diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & A & & \\
\downarrow & & \downarrow & & \\
B & \longrightarrow & A \oplus B & \longrightarrow & A \\
& & \downarrow & & \downarrow \\
& & B & \longrightarrow & 0
\end{array}$$

the left-hand square is a pushout and the right-hand square is a pullback.

The class of split exact sequences actually defines an exact structure, *split* on any additive category \mathcal{E} . The above remarks imply the following proposition.

Proposition 2.4. *Let $(\mathcal{E}, \mathcal{Q})$ be an exact category. Then the identity functor $\text{id}_{\mathcal{E}}$ is an exact functor $(\mathcal{E}, \text{split}) \rightarrow (\mathcal{E}, \mathcal{Q})$.*

At the other extreme we have quasi-abelian exact structures.

Definition 2.5. *An additive category \mathcal{E} with all kernels and cokernels is said to be **quasi-abelian** if the class qac of all kernel-cokernel pairs forms an exact structure on \mathcal{E} .*

The following is then tautological.

Proposition 2.6. *Let \mathcal{E} be a quasi-abelian category, and let \mathcal{Q} be a class of kernel-cokernel pairs on \mathcal{E} such that $(\mathcal{E}, \mathcal{Q})$ is an exact category. Then the identity functor $\text{id}_{\mathcal{E}}$ is an exact functor $(\mathcal{E}, \mathcal{Q}) \rightarrow (\mathcal{E}, \text{qac})$.*

We will study quasi-abelian structures in more detail later. For now let us note that abelian categories are quasi-abelian. In an abelian category all monics are kernels of their cokernels, and all epics are cokernels of their kernels. It therefore trivially follows that both classes are closed under composition. It is also clear that both classes contain all isomorphisms. It is a standard exercise that in an abelian category, monomorphisms are pushout-stable and epimorphisms are pullback-stable. See for example [Fre64] Theorem 2.54. Let us now record some basic results about exact categories which will prove useful.

Proposition 2.7. *Let $h : X \rightarrow Y$ and $f : A \rightarrow Y$ be morphisms in an additive category \mathcal{E} . Suppose $g : K \rightarrow X$ is a kernel of h and that the following diagram*

$$\begin{array}{ccc}
A' & \xrightarrow{k} & X \\
\downarrow q & & \downarrow h \\
A & \xrightarrow{f} & Y
\end{array}$$

is a pullback. Then there is a commutative diagram

$$\begin{array}{ccc}
K & \xlongequal{\quad} & K \\
\downarrow g' & & \downarrow g \\
A' & \xrightarrow{k} & X
\end{array}$$

with $g' : K \rightarrow A'$ being a kernel of q .

Dually, if $f : Y \rightarrow A$ and $h : Y \rightarrow X$ are morphisms in an additive category \mathcal{E} , $g : X \rightarrow C$ is a cokernel of h and the following diagram

$$\begin{array}{ccc}
Y & \xrightarrow{f} & A \\
\downarrow h & & \downarrow k \\
X & \xrightarrow{q} & Y'
\end{array}$$

is a push out, then there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & Y' \\ \downarrow g & & \downarrow g' \\ C & \xlongequal{\quad} & C \end{array}$$

where $g' : Y' \rightarrow C$ is a cokernel of k .

Proof. Consider the zero morphism $0 : K \rightarrow A$. Then $h \circ g = 0 = f \circ 0$. So there is a unique morphism $g' : K \rightarrow A'$ such that $q \circ g' = 0$ and $k \circ g' = g$. This gives the commutative diagram. It remains to show that g' is a kernel of q . Let $r : Z \rightarrow A'$ be a morphism such that $q \circ r = 0$. Then $h \circ k \circ r = f \circ q \circ r = 0$. Thus there is a unique map $t : Z \rightarrow K$ such that $g \circ t = k \circ r$. But then $k \circ g' \circ t = k \circ r$. Moreover, $q \circ g' \circ t = 0$. This means $g' \circ t = r$. If t' is another map such that $g' \circ t' = r$, then $g \circ t' = k \circ g' \circ t' = k \circ r$. But t was the unique map such that $g \circ t = k \circ r$, so $t = t'$. Hence $g' : K \rightarrow A'$ is a kernel of $q : A' \rightarrow A$. The second part is dual. \square

In fact in an exact category we actually have the following useful result.

Proposition 2.8. *Let*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow f & & \downarrow f' \\ A' & \xrightarrow{i'} & B' \end{array}$$

be a commutative diagram in which the horizontal morphisms are admissible monics. Then the following are equivalent

(1) *The square above is a push-out.*

(2) *The sequence*

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} i \\ -f \end{pmatrix}} B \oplus A' \xrightarrow{(f' \ i')} B' \longrightarrow 0$$

is short exact.

(3) *The square above is bicartesian.*

(4) *The square is part of a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{p} & C \\ \downarrow f & & \downarrow f' & & \parallel \\ A' & \xrightarrow{i'} & B' & \xrightarrow{p'} & C \end{array}$$

with short exact rows.

Proof. See [Büh10] Proposition 2.12. \square

For technical reasons, unless stated otherwise we will assume from now on that all exact categories are **weakly idempotent complete**. This means that every retraction has a kernel, or equivalently, that every coretraction has a cokernel. Note that the condition is self-dual. Quasi-abelian categories are in particular weakly idempotent complete. In weakly idempotent complete exact categories, we then have the following useful result, often called the **Obscure Axiom**.

Proposition 2.9 (The Obscure Axiom). *(1) Suppose that $i : A \rightarrow B$ is a morphism. If there exists a morphism $j : B \rightarrow C$ such that the composite $ji : A \rightarrow C$ is an admissible monic, then i is an*

admissible monic.

- (2) Suppose that $i : A \rightarrow B$ is a morphism. If there exists a morphism $j : C \rightarrow A$ such that $i \circ j$ is an admissible epic, then i is an admissible epic.

Proof. See [Büh10] Proposition 2.16. □

2.0.3. *The Embedding Theorem.* Let $(\mathcal{E}, \mathcal{Q})$ be an exact category. Let \mathcal{F} be a full subcategory of \mathcal{E} . Suppose that \mathcal{F} is closed under extensions, that is if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence in $(\mathcal{E}, \mathcal{Q})$ with A and C objects of \mathcal{F} , then B is an object of \mathcal{F} as well. Let $\mathcal{Q}_{\mathcal{F}}$ consist of those kernel-cokernel pairs $(i : A \rightarrow B, q : B \rightarrow C)$ in \mathcal{F} which when regarded as pairs of morphisms in \mathcal{E} are kernel-cokernel pairs in \mathcal{Q} . It is then straight-forward to show ([BMSG⁺10]) that $(\mathcal{F}, \mathcal{Q}_{\mathcal{F}})$ is an exact subcategory of $(\mathcal{E}, \mathcal{Q})$. It turns out that any small exact category can be obtained as a full subcategory of an abelian category which is closed under extensions. This is the main content of the Quillen Embedding Theorem which provides an invaluable tool for studying exact categories.

Theorem 2.10 (The Quillen Embedding Theorem). *Let \mathcal{E} be a small exact category. Then there is an abelian category $\mathcal{A}(\mathcal{E})$ and a fully faithful additive functor $I : \mathcal{E} \rightarrow \mathcal{A}(\mathcal{E})$ which is exact, reflects exactness, and preserves all kernels. Moreover the essential image of I is closed under extensions. $\mathcal{A}(\mathcal{E})$ may be chosen to be the category of left-exact functors $\mathcal{E} \rightarrow \mathbf{Ab}$. If in addition \mathcal{E} is weakly idempotent complete then a morphism $f : E \rightarrow F$ in \mathcal{E} is an admissible epic if and only if $I(f)$ is an epic in $\mathcal{A}(\mathcal{E})$.*

Proof. See Appendix A in [Büh10]. □

Definition 2.11. We call an embedding $I : \mathcal{E} \rightarrow \mathcal{A}$ of an exact category into an abelian category a **left abelianization** of \mathcal{E} if

- (1) I is fully faithful.
- (2) I is exact.
- (3) I reflects exactness.
- (4) The essential image of I is closed under extensions.
- (5) I preserves all kernels which exist.
- (6) If f is a morphism in \mathcal{E} , then f is an admissible epic if and only if $I(f)$ is an epic.

In particular, Theorem 2.10 says that any small exact category admits a left abelianization. Dually there is a notion of a right abelianization

Definition 2.12. We call an embedding $I : \mathcal{E} \rightarrow \mathcal{A}$ of an exact category into an abelian category a **right abelianization** of \mathcal{E} if

- (1) I is fully faithful.
- (2) I is exact.
- (3) I reflects exactness.
- (4) The essential image of I is closed under extensions.
- (5) I preserves all cokernels which exist.

(6) If f is a morphism in \mathcal{E} , then f is an admissible monic if and only if $I(f)$ is a monic.

Remark 2.13. It is clear that right abelianizations of small exact categories exist. Indeed, if $\mathcal{E}^{op} \rightarrow \mathcal{A}$ is a left-abelianization of \mathcal{E}^{op} , then $\mathcal{E} \rightarrow \mathcal{A}^{op}$ is a right-abelianization of \mathcal{E} .

From now on unless stated otherwise, we will assume that all our categories are equipped with both a left and right abelianisation. This is not just true of small exact categories. For example any quasi-abelian category has both a left and right abelianisation.

2.1. Notions of Acyclicity. In a general exact category, arbitrary kernels and cokernels may not exist. Therefore it is not in general possible even to write down candidates for the homology objects of a chain complex. Even if all kernels and cokernels do exist, then there are multiple candidates for the homology which are not isomorphic in general. For example, given a null sequence

$$\Gamma = E \xrightarrow{f} F \xrightarrow{g} G$$

i.e. $g \circ f = 0$, one could consider both $\text{Coker}(\text{Im}(f) \rightarrow \text{Ker}(g))$ and $\text{Im}(\text{Ker}(g) \rightarrow \text{Coker}(f))$. In an abelian category these are isomorphic, but for general additive categories this is not the case. Despite these ambiguities, there are still various useful notions of acyclicity in exact categories, which we discuss below. First let us define several classes of morphisms.

Definition 2.14. A morphism $f : E \rightarrow F$ in an exact category is said to be

(1) **weakly left admissible** if it has a kernel and the map

$$\text{Ker}(f) \rightarrow E$$

is admissible.

(2) **weakly right admissible** if it has a cokernel, and the map

$$F \rightarrow \text{Coker}(f)$$

is admissible.

(3) **weakly admissible** if it is both weakly left admissible and weakly right admissible.

The following characterisation of weakly admissible morphisms is immediate.

Proposition 2.15. A morphism $f : E \rightarrow F$ in an exact category \mathcal{E} is weakly admissible if and only if it admits a decomposition

$$\begin{array}{ccccc} & & E & \xrightarrow{f} & F \\ & \nearrow & & & \searrow \\ \text{Ker}(f) & & & & \text{Coker}(f) \\ & \searrow & & \nearrow & \\ & & \text{Coim}(f) \xrightarrow{\hat{f}} \text{Im}(f) & & \end{array}$$

where the sequences

$$\text{Ker}(f) \rightarrowtail E \twoheadrightarrow \text{Coim}(f)$$

and

$$\text{Im}(f) \rightarrowtail F \twoheadrightarrow \text{Coker}(f)$$

are short exact.

Definition 2.16. Let f be a morphism in exact category. Then f is said to be **admissible** if it is weakly admissible and the map $\text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism.

Remark 2.17. Admissible epimorphisms and admissible monomorphisms are admissible morphisms in the sense above.

In diagrams we shall denote admissible morphisms by \twoheadrightarrow .

This is not how admissible morphisms are usually defined (see e.g. [Büh10]). However the notions are equivalent:

Proposition 2.18. *Let $f : E \rightarrow F$ be a morphism in an exact category \mathcal{E} . Then the following are equivalent.*

(1) *f is admissible.*

(2) *f admits a decomposition*

$$E \twoheadrightarrow I \rightarrowtail F$$

(3) *There is a commutative diagram*

$$\begin{array}{ccccc} & E & \xrightarrow{f} & F & \\ \nearrow & & & & \searrow \\ \text{Ker}f & & I & & \text{Coker}f \end{array}$$

where the sequences

$$\text{Ker}f \rightarrowtail E \twoheadrightarrow I$$

and

$$I \rightarrowtail F \twoheadrightarrow \text{Coker}(f)$$

are short exact.

Proof. 1 and 3 are clearly equivalent thanks to Proposition 2.15. Also $3 \Rightarrow 2$ trivially. Let us show that $2 \Rightarrow 1$. Since $I \rightarrowtail F$ is an admissible monic, the kernel of f exists, and coincides with the kernel of $E \twoheadrightarrow I$. Hence $\text{Ker}(f) \rightarrow E$ is an admissible monic and in particular $E \twoheadrightarrow I$ is a coimage of f . Dually, the cokernel of f exists, it coincides with the cokernel of $G \rightarrowtail F$, and $I \rightarrowtail F$ is an image of f . \square

Corollary 2.19. *A morphism $f : E \rightarrow F$ in an exact category is an isomorphism if and only if it is both an admissible epic and an admissible monic.*

Proof. Axiomatically an isomorphism is both an admissible monic and an admissible epic.

Conversely, suppose f is both an admissible monic and an admissible epic. Since it is an admissible monic the map $E \rightarrow \text{Coim}(f)$ is an isomorphism. Since it is an admissible epic the map $\text{Im}(f) \rightarrow F$ is an isomorphism. Since f is admissible the map $\text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism. The claim now follows from the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow \sim & & \sim \uparrow \\ \text{Coim}(f) & \xrightarrow{\sim} & \text{Im}(f) \end{array}$$

\square

We are now ready to introduce our various notions of acyclic sequences.

Definition 2.20. *A null-sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is said to be

(1) **weakly acyclic** if f is weakly right admissible, g has a kernel, and the natural map $\text{Im}(f) \rightarrow \text{Ker}(g)$ is an isomorphism.

- (2) **weakly coacyclic** if g is weakly left admissible, f has a cokernel, and the natural map $\text{Coker}(f) \rightarrow \text{Coker}(g)$ is an isomorphism.
- (3) **admissibly acyclic** if it is weakly acyclic and f is admissible,
- (4) **admissibly coacyclic** if it is weakly coacyclic and g is admissible
- (5) **admissible** if both f and g are admissible.
- (6) **acyclic** if it is both admissibly acyclic and admissibly coacyclic.

Remark 2.21. If a null sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is weakly acyclic then g is automatically weakly left admissible.

Definition 2.22. A complex

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \longrightarrow X_0$$

is said to be weakly acyclic/ weakly coacyclic/ admissibly acyclic/ admissibly coacyclic/ admissible/ acyclic if for each $1 \leq i \leq n-1$ each sequence

$$X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1}$$

is weakly acyclic/ weakly coacyclic/ admissibly acyclic/ admissibly coacyclic/ admissible/ acyclic.

Let us now set up some tools for determining whether a complex is acyclic.

We can partially test acyclicity by passing to a left abelianisation:

Proposition 2.23. Let $I : \mathcal{E} \rightarrow \mathcal{A}$ be a left abelianization of \mathcal{E} .

(1) If

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \longrightarrow X_0$$

is admissibly acyclic in \mathcal{E} then

$$I(X_n) \xrightarrow{I(f_n)} I(X_{n-1}) \xrightarrow{I(f_{n-1})} \dots \longrightarrow I(X_0)$$

is exact in \mathcal{A} .

(2) If f_i is weakly admissible for $2 \leq i \leq n$ and

$$I(X_n) \xrightarrow{I(f_n)} I(X_{n-1}) \xrightarrow{I(f_{n-1})} \dots \longrightarrow I(X_0)$$

is exact, then

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \longrightarrow X_0$$

is admissibly acyclic.

(3) If f_i is weakly left admissible for $1 \leq i \leq n-1$ and

$$I(X_n) \xrightarrow{I(f_n)} I(X_{n-1}) \xrightarrow{I(f_{n-1})} \dots \longrightarrow I(X_0)$$

is exact in \mathcal{A} , then

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \longrightarrow X_0$$

is admissibly acyclic.

Proof. Clearly it is sufficient to prove the claims for sequences

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

- (1) Suppose the above sequence is admissibly acyclic. Since f is admissible I preserves $\text{Im}(f)$. By assumption I preserves all kernels. Hence

$$I(X) \xrightarrow{I(f)} I(Y) \xrightarrow{I(g)} I(Z)$$

is exact.

- (2) Suppose now that

$$I(X) \xrightarrow{I(f)} I(Y) \xrightarrow{I(g)} I(Z)$$

is exact and that f is weakly admissible. Since I preserves all kernels, and cokernels of admissible morphisms, we have $I(\text{Coim}(f)) \cong \text{Coim}I(f)$. Now

$$\text{Coim}I(f) \cong \text{Im}I(f) \cong \text{Ker}I(g)$$

Since I is fully faithful, $\text{Coim}(f)$ is a kernel of g . Finally, note that we have a factorisation of $\text{Coim}(f) \rightarrow \text{Ker}(g)$

$$\text{Coim}(f) \rightarrow \text{Im}(f) \rightarrow \text{Ker}(g)$$

By Proposition 2.9 $\text{Im}(f) \rightarrow \text{Ker}(g)$ is also an (admissible) epic. By Corollary 2.19 it is an isomorphism. Therefore $\text{Coim}(f) \rightarrow \text{Im}(f)$ is as well. By Proposition 2.18 we are done.

- (3) We can factor f as

$$X \xrightarrow{f'} \text{Ker}(g) \longrightarrow Y$$

with $\text{Ker}(g) \rightarrow Y$ an admissible monic. We need to show f' is an admissible epic. Since I preserves kernels, it sends the diagram above to

$$I(X) \xrightarrow{I(f')} \text{Ker}I(g) \longrightarrow I(Y)$$

Since

$$I(X) \xrightarrow{I(f)} I(Y) \xrightarrow{I(g)} I(Z)$$

is exact, $I(f')$ is an epic. thus f' is an admissible epic, and we are done. □

Part 1) of the above proposition says that the functor I is admissibly exact. This is a stronger notion than exactness. It will be useful in later contexts, so we make a definition.

Definition 2.24. A functor $F : \mathcal{E} \rightarrow \mathcal{F}$ between exact categories is said to be **admissibly (co)exact** if for any admissibly (co)acyclic sequence

$$X \rightarrow Y \rightarrow Z$$

in \mathcal{E} , the sequence

$$F(X) \rightarrow F(Y) \rightarrow F(Z)$$

is admissibly (co)acyclic. A functor which is both admissibly exact and admissibly coexact is said to be **strongly exact**.

Moreover, the proof of Part 1) also gives the following result.

Proposition 2.25. Let $F : \mathcal{E} \rightarrow \mathcal{D}$ be an exact functor which preserves kernels. Then F is admissibly exact.

Example 2.26. It is easy to show that taking finite direct sums is a strongly exact functor. Indeed being both a limit and a colimit, this functor commutes with all limits and colimits.

Although the functor I reflects short exact sequences, it need not in general reflect acyclicity of unbounded complexes. However it does for a certain nice class of complexes.

Definition 2.27. A complex X_\bullet in an exact category is said to be **good** if for each n there is $m < n$ such that d_m has a kernel. X_\bullet is said to be **cogood** if for each n there is $m > n$ such that d_m has a cokernel.

Example 2.28. Bounded below complexes are good.

We will frequently use the following trick for good complexes.

Proposition 2.29. Let X_\bullet be a good complex in an exact category. Suppose that for any n such that d_n^X has a kernel, the induced map

$$d'_{n+1} : X_{n+1} \rightarrow Z_n X$$

is an admissible epic. Then X_\bullet is acyclic.

Proof. Suppose d_m has a kernel. By assumption d_{m+1} factors as

$$X_{m+1} \rightarrow Z_m X \rightarrow X_m$$

A priori $Z_m X \rightarrow X_m$ is not admissible. However it is a monomorphism. Therefore, since $X_{m+1} \rightarrow Z_m X$ is admissible its kernel exists and it coincides with the kernel $Z_{m+1} X$ of d_{m+1} . Since $X_{m+1} \rightarrow Z_m X$ is admissible it is in particular weakly left admissible. Therefore d_{m+1} is also weakly left admissible. Now consider d_{m+2} . By assumption it factors as

$$d_{m+2} : X_{m+2} \rightarrow Z_{m+1} X \rightarrow X_{m+1}$$

Thus d_{m+2} is an admissible morphism whose image is $Z_{m+1} X$. An easy induction then shows that X_\bullet is acyclic. \square

Since I preserves kernels and reflects admissible epimorphisms, Proposition 2.29 gives the following.

Corollary 2.30. Let (X_\bullet, d_\bullet) be a complex in \mathcal{E} . Let $I : \mathcal{E} \rightarrow \mathcal{A}$ be a left abelianisation of \mathcal{E} . Suppose X_\bullet is good. Then X_\bullet is acyclic if and only if $I(X_\bullet)$ is.

Proof. Suppose $I(X_\bullet)$ is acyclic, and d_n^X has a kernel $Z_n X$. By assumption $I(d'_{n+1}) : I(X_{n+1}) \rightarrow Z_n I(X) = I(Z_n X)$ is an epimorphism. Thus $d'_{n+1} : X_{n+1} \rightarrow Z_n X$ is an admissible epimorphism. \square

2.2. Homotopies and Quasi-Isomorphisms. Let us now discuss homological properties of maps between complexes.

Definition 2.31. A **homotopy** between morphisms of chain complexes $f_\bullet, g_\bullet : K_\bullet \rightarrow L_\bullet$ is a collection of morphisms $D_i : A_i \rightarrow B_{i+1}$ such that

$$f_i - g_i = D_{i-1} \circ d_i^K + d_{i+1}^L \circ D_i$$

We then say $f_\bullet \sim g_\bullet$.

Definition 2.32. Two complexes K_\bullet and L_\bullet are said to be **homotopy equivalent** if there are maps $g : K_\bullet \rightarrow L_\bullet$ and $f : L_\bullet \rightarrow K_\bullet$ such that $f \circ g \sim \text{id}_{K_\bullet}$ and $g \circ f \sim \text{id}_{L_\bullet}$.

If

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ \downarrow & & \downarrow \alpha & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

is a diagram with the top and bottom row being null-sequences, we will also say that it is homotopic to zero if there are two maps $D : B \rightarrow X$ and $D' : C \rightarrow Y$ such $\alpha = f \circ D - D' \circ q$.

We can use homotopies in an exact category to test for acyclicity.

Proposition 2.33. *Let \mathcal{E} be an exact category, and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a null sequence. Suppose that g has a kernel. Then the induced map $f' : X \rightarrow \text{Ker}(g)$ is an admissible epimorphism if and only if there is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ \downarrow & & \downarrow \alpha & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

which is homotopic to zero, and such that the induced map $\tilde{\alpha} : \text{Ker}(q) \rightarrow \text{Ker}(g)$ is an admissible epic.

Proof. Suppose that g has a kernel and that the induced map $f' : X \rightarrow \text{Ker}(g)$ is an admissible epimorphism. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

By assumption the induced map $\tilde{f} : X \rightarrow \text{Ker}(g)$ is an admissible epic. Moreover the diagram is clearly homotopic to 0 via the maps $D = \text{id} : X \rightarrow X$ and $D' = 0 : 0 \rightarrow Y$.

Conversely suppose we have a diagram

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ \downarrow & \nearrow D & \downarrow \alpha & \nearrow D' & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

such that g has a kernel and $\alpha = f \circ D - D' \circ q$. We have the factorisation of f

$$X \xrightarrow{\tilde{f}} \text{Ker}(g) \longrightarrow Y$$

Moreover, $\tilde{\alpha} = \tilde{f} \circ D|_{\text{Ker}(q)}$. By Proposition 2.9 \tilde{f} is an admissible epic. □

Corollary 2.34. *Let \mathcal{E} be an exact category, and let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a null sequence. The sequence is admissibly acyclic if and only if g is weakly left admissible and there is a diagram

$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ \downarrow & & \downarrow \alpha & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

which is homotopic to zero, and such that the induced map $\tilde{\alpha} : \text{Ker}(q) \rightarrow \text{Ker}(g)$ is an admissible epic.

Proof. Suppose the sequence is admissibly acyclic. By Remark 2.21 g is weakly left admissible.

For the converse, note that by Proposition 2.33 and the fact that $\text{Ker}(g) \rightarrow Y$ is admissible, we have a decomposition of f

$$X \twoheadrightarrow \text{Ker}(g) \rightarrowtail Y$$

By Proposition 2.18 f is an admissible morphism whose image is $\text{Ker}(g)$. □

We can also test split exactness by looking at homotopy.

Proposition 2.35. *Let \mathcal{E} be an exact category, and let*

$$\Gamma := X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a null-sequence. The sequence is admissibly acyclic in the split exact structure if and only if g is weakly left admissible and the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow id_X & & \downarrow id_Y & & \downarrow id_Z \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

is homotopic to zero.

Proof. Suppose the diagram is homotopic to the zero. If we can show that g is also weakly left admissible in the split exact structure, then the claim follows from Corollary 2.34. By Corollary 2.34 we already know that the sequence is admissibly acyclic, so $\text{Im}(f) \cong \text{Ker}(g)$. Let $D : Y \rightarrow X$ and $D' : Z \rightarrow Y$ be maps such that $id_Y = f \circ D - D' \circ g$. The map $f \circ D : Y \rightarrow Y$ factors as

$$Y \xrightarrow{(f \circ D)} \text{Im}(f) \xrightarrow{i} Y$$

where i is the inclusion. But

$$f \circ D \circ i = f \circ D \circ i - D \circ g \circ i = i$$

since $g \circ i = 0$. It follows that $(f \circ D) \circ i = \text{Id}_{\text{Im}(f)}$. This implies that the map $\text{Ker}(g) \cong \text{Im}(f) \rightarrow Y$ is split, and so is an admissible monic in the split exact structure. \square

Corollary 2.36. *Let X_\bullet be a good complex.*

- (1) *X_\bullet is acyclic whenever there is a complex Y_\bullet , a morphism of complexes $f_\bullet : Y_\bullet \rightarrow X_\bullet$ which is homotopic to 0, and such that the induced maps $\tilde{f}_n : \text{Ker}(d_n^Y) \rightarrow \text{Ker}(d_n^X)$ are admissible epimorphisms.*
- (2) *X_\bullet is split exact whenever id_{X_\bullet} is homotopic to 0.*

Proof. The first assertion follows from Proposition 2.29 and Proposition 2.33. For the second assertion note that X_\bullet is acyclic by the first. In particular each

$$X_{n+1} \rightarrow X_n \rightarrow X_{n-1}$$

is acyclic, and $X_n \rightarrow X_{n-1}$ is (weakly left) admissible. Thus we may use Proposition 2.35. \square

2.2.1. Quasi-Isomorphisms. Recall that in an abelian category a map of complexes induces a map on homology. The map is said to be a quasi-isomorphism if the induced map on homology is an isomorphism. Quasi-isomorphisms can also be characterised in terms of their mapping cone. A map of chain complexes in an abelian category is a quasi-isomorphism if and only if its mapping cone is acyclic. As remarked previously, in an exact category we cannot in general define the homology of a complex. However the construction of the mapping cone makes sense in any additive category. By the previous remarks, the following definition is sensible.

Definition 2.37. *Let \mathcal{E} be an exact category. A map $f_\bullet : X_\bullet \rightarrow Y_\bullet$ of complexes of \mathcal{E} is said to be a **quasi-isomorphism** if $\text{cone}(f_\bullet)$ is acyclic.*

Proposition 2.38. *Homotopy equivalences are quasi-isomorphisms.*

Proof. See [Büh10] Proposition 10.9. \square

The next proposition is an immediate consequence of Corollary 2.30.

Proposition 2.39. *Let $I : \mathcal{E} \rightarrow \mathcal{A}$ be a left abelianisation of an exact category \mathcal{E} . Let $f_\bullet : X_\bullet \rightarrow Y_\bullet$ be a morphism of complexes. Suppose $\text{cone}(f)_m$ is good. Then f is a quasi-isomorphism if and only if $I(f)$ is.*

Remark 2.40. *As for abelian categories, one can define the derived category $D_*(\mathcal{E})$ of an exact category \mathcal{E} by localizing $\text{Ch}_*(\mathcal{E})$ at the quasi-isomorphisms. For details see for example [Büh10].*

2.3. Ext Groups. In order to study cotorsion pairs in exact categories in Section 3, we will need to first introduce Ext groups in exact categories. Recall for an abelian category \mathcal{A} one can define the groups $\text{Ext}^n(A, B)$ for any pair of objects $A, B \in \mathcal{A}$ regardless of whether \mathcal{A} has enough projectives. The elements are Yoneda equivalence classes of n -extensions and the binary operation is the Baer sum. Moreover for objects A, C, D and integers m, n there are bilinear maps

$$\phi_{n,m} : \text{Ext}^n(A, C) \times \text{Ext}^m(C, D) \rightarrow \text{Ext}^{n+m}(A, D)$$

All of this works mutatis mutandis for weakly idempotent complete exact categories \mathcal{E} once we make the following definition.

Definition 2.41. *Let A and C be objects of \mathcal{E} , and let $n > 0$ be an integer. An n -**fold extension** of A by C is an acyclic sequence*

$$0 \rightarrow C \rightarrow E_n \rightarrow E_{n-1} \rightarrow \dots \rightarrow E_1 \rightarrow A \rightarrow 0$$

Moreover in this greater generality we still get the long-exact sequences on Ext groups. let

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

be short exact. We define a map $\delta_n : \text{Ext}^n(X, C) \rightarrow \text{Ext}^{n+1}(Z, C)$ for each $n > 0$ as follows:

Recall we have a bilinear map

$$\phi_{1,n} : \text{Ext}^1(Z, X) \times \text{Ext}^n(X, C) \rightarrow \text{Ext}^{n+1}(Z, C)$$

Let K be the class in $\text{Ext}^1(Z, X)$ representing

$$0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0$$

We set $\delta_n = \phi_{1,n}(K, -)$.

Theorem 2.42. *Let*

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

be a short exact sequence in \mathcal{E} , and let A be an object of \mathcal{E} . Then there are exact sequences

$$0 \longrightarrow \text{Hom}(Z, A) \longrightarrow \text{Hom}(Y, A) \longrightarrow \text{Hom}(X, A) \longrightarrow \text{Ext}^1(Z, A) \longrightarrow \dots \longrightarrow$$

$$\text{Ext}^n(X, A) \longrightarrow \text{Ext}^{n+1}(Z, A) \longrightarrow \text{Ext}^{n+1}(Y, A) \longrightarrow \text{Ext}^{n+1}(X, A) \longrightarrow \dots$$

and

$$0 \longrightarrow \text{Hom}(A, X) \longrightarrow \text{Hom}(A, Y) \longrightarrow \text{Hom}(A, Z) \longrightarrow \text{Ext}^1(A, X) \longrightarrow \dots \longrightarrow$$

$$\text{Ext}^n(A, Z) \longrightarrow \text{Ext}^{n+1}(A, X) \longrightarrow \text{Ext}^{n+1}(A, Y) \longrightarrow \text{Ext}^{n+1}(A, Z) \longrightarrow \dots$$

All the proofs for the above facts work as the abelian case. The interested reader can adapt the relevant proofs in [Buc59] for example.

The first ext group $\text{Ext}^1(A, B)$ can also be computed by passing to a left abelianization. More generally we have the following:

Proposition 2.43. *Let \mathcal{E} and \mathcal{F} be exact categories. Let $F : \mathcal{E} \rightarrow \mathcal{F}$ be a fully faithful exact functor which reflects exactness. Suppose that the essential image of \mathcal{E} is closed under extensions. Then F induces a natural isomorphism of abelian groups*

$$\text{Ext}_{\mathcal{E}}^1(-, -) \cong \text{Ext}_{\mathcal{F}}^1(F(-), F(-))$$

Proof. We define a map $\text{Ext}_{\mathcal{E}}(X, Y) \rightarrow \text{Ext}_{\mathcal{F}}(F(X), F(Y))$ by sending a short exact sequence

$$0 \rightarrow Y \rightarrow P \rightarrow X \rightarrow 0$$

to

$$0 \rightarrow F(Y) \rightarrow F(P) \rightarrow F(X) \rightarrow 0$$

This map is easily seen to be well-defined. Let us show that this map is onto. Suppose

$$0 \longrightarrow F(Y) \xrightarrow{f} Q \xrightarrow{g} F(X) \longrightarrow 0$$

is exact. Since the essential image of F is closed under extensions there is some object $P \in \mathcal{E}$ and an isomorphism $p : Q \xrightarrow{\sim} F(P)$. Therefore we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(Y) & \xrightarrow{f} & Q & \xrightarrow{g} & F(X) \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow p & & \downarrow \text{id} \\ 0 & \longrightarrow & F(Y) & \xrightarrow{p \circ f} & F(P) & \xrightarrow{g \circ p^{-1}} & F(X) \longrightarrow 0 \end{array}$$

For posterity, we also note that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(Y) & \xrightarrow{f} & Q & \xrightarrow{g} & F(X) \longrightarrow 0 \\ & & \uparrow \text{id} & & \uparrow p^{-1} & & \uparrow \text{id} \\ 0 & \longrightarrow & F(Y) & \xrightarrow{p \circ f} & F(P) & \xrightarrow{g \circ p^{-1}} & F(X) \longrightarrow 0 \end{array}$$

Now the bottom row is exact. Moreover, since F is fully faithful there is a unique $f' : Y \rightarrow P$ and a unique $g' : P \rightarrow X$ such that $p \circ f = F(f')$ and $g \circ p^{-1} = F(g')$. Since F reflects exactness

$$0 \longrightarrow Y \xrightarrow{f'} P \xrightarrow{g'} X \longrightarrow 0$$

is exact in \mathcal{A} .

By construction there is a pre-equivalence

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(Y) & \xrightarrow{f} & Q & \xrightarrow{g} & F(X) \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow p & & \downarrow \text{id} \\ 0 & \longrightarrow & F(Y) & \xrightarrow{p \circ f} & F(P) & \xrightarrow{g \circ p^{-1}} & F(X) \longrightarrow 0 \end{array}$$

Hence the map is surjective.

Next let us show that the map is injective. Suppose that the short exact sequences

$$A = 0 \longrightarrow F(Y) \longrightarrow F(P) \longrightarrow F(X) \longrightarrow 0$$

$$B = 0 \longrightarrow F(Y) \longrightarrow F(P') \longrightarrow F(X) \longrightarrow 0$$

are Yoneda equivalent. Thus there are short exact sequences

$$C_i = 0 \longrightarrow F(Y) \longrightarrow Q_i \longrightarrow F(X) \longrightarrow 0$$

for $0 \leq i \leq n$, with $C_0 = A$, $C_n = B$, and either C_i is pre-equivalent to C_{i+1} , or C_{i+1} is pre-equivalent to C_i for $0 \leq i \leq n-1$. However, it follows from the above discussion that we may assume

$$C_i = 0 \longrightarrow F(Y) \longrightarrow F(P_i) \longrightarrow F(X) \longrightarrow 0$$

It is therefore enough to show that if we have a Yoneda pre-equivalence

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(Y) & \xrightarrow{F(f)} & F(Q) & \xrightarrow{F(g)} & F(X) \longrightarrow 0 \\ & & \downarrow id & & \downarrow p & & \downarrow id \\ 0 & \longrightarrow & F(Y) & \xrightarrow{F(h)} & F(P) & \xrightarrow{F(k)} & F(X) \longrightarrow 0 \end{array}$$

then there is a map $q : Q \rightarrow P$ in \mathcal{E} , with $F(q) = p$ so that the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \xrightarrow{f} & Q & \xrightarrow{g} & X \longrightarrow 0 \\ & & \downarrow id & & \downarrow q & & \downarrow id \\ 0 & \longrightarrow & Y & \xrightarrow{h} & P & \xrightarrow{k} & X \longrightarrow 0 \end{array}$$

This follows from the fact that F is full and faithful.

It remains to show that we have a map of abelian groups. However this follows from Proposition 2.8. \square

Remark 2.44. *In the above we make the implicit assumption that each $\text{Ext}^n(A, B)$ is a set. This always holds for exact categories with enough projectives, which can be seen from the discussion in the following section.*

2.4. Projective Objects and Resolutions in Exact Categories. At this point we recall the notion of a projective object in an exact category, and mention how they relate to the Ext functor.

Definition 2.45. *An object P in an exact category \mathcal{E} is said to be **projective** if the functor $\text{Hom}(P, -) : \mathcal{E} \rightarrow \mathbf{Ab}$ is exact.*

Remark 2.46. *By Proposition 2.25, for any projective object P the functor $\text{Hom}(P, -)$ is admissibly exact.*

Example 2.47. *In the split exact structure every object is projective.*

As in the abelian case one has the following result.

Proposition 2.48. *The following are equivalent.*

- (1) P is projective.
- (2) *Given a map $f : P \rightarrow C$ and an admissible epic $e : B \rightarrow C$, there is a morphism $g : P \rightarrow B$ such that the following diagram commutes*

$$\begin{array}{ccc} & & B \\ & \nearrow g & \downarrow e \\ P & \xrightarrow{f} & C \end{array}$$

- (3) *Any admissible epic with codomain P splits.*
- (4) $\text{Ext}^1(P, A)$ vanishes for any object A .
- (5) $\text{Ext}^n(P, A)$ vanishes for any object A and any $n \geq 1$.

We will need some results about projective resolutions in exact categories later.

2.4.1. Bounded Resolutions.

Definition 2.49. An exact category \mathcal{E} is said to **have enough projectives** if for any object X of \mathcal{E} , there is a projective object P and an admissible epimorphism $P \twoheadrightarrow X$.

Lemma 2.50. Let \mathcal{P} be a subclass of $\mathbf{Ob}(\mathcal{E})$. Assume that for any object E of \mathcal{E} there is an object $P \in \mathcal{P}$ and an admissible epimorphism $P \twoheadrightarrow E$. Then, for any bounded below complex E of $\mathbf{Ch}_+(\mathcal{E})$, there is a bounded below complex P whose entries are objects of \mathcal{P} , and a quasi-isomorphism

$$u : P \rightarrow E$$

where each $u_k : P_k \rightarrow E_k$ is an admissible epimorphism. Moreover, this construction can be made functorial if the choice of admissible epimorphism $P \twoheadrightarrow E$ can be made functorial.

Proof. This is proved in [Büh10] for the case that \mathcal{P} is the class of projectives in an exact category with enough projectives. However the proof goes through the same. \square

Lemma 2.51. Let A, B be objects in an exact category \mathcal{E} . Let $f : A \rightarrow B$ be a morphism. Let P_\bullet be a complex with $P_{-1} = A, P_n = 0$ for $n < -1$ and P_n projective for $n > 0$. Also let Q_\bullet be an acyclic complex with $Q_{-1} = B$ and $Q_n = 0$ for $n < -1$. Then there is a chain map $f_\bullet : P_\bullet \rightarrow Q_\bullet$ with $f_{-1} = f$. Moreover, f_\bullet is unique up to homotopy.

Proof. See [Büh10] Theorem 12.4. \square

As in the abelian case one can define derived functors between derived categories of exact categories. There are also notions of adapted classes for functors. Proposition 2.48 and Lemma 2.50 essentially say that as in the abelian case, if a category \mathcal{E} has enough projectives, then the class of projective objects is adapted to the functor $\mathrm{Hom}(-, A) : \mathcal{E}^{\mathrm{op}} \rightarrow \mathbf{Ab}$. It can be shown that $R^n \mathrm{Hom}(-, A) := H_n(R\mathrm{Hom}(-, A)) \cong \mathrm{Ext}^n(-, A)$.

2.4.2. Unbounded Resolutions. When dealing with the model structures on unbounded chain complexes, we will also need to have unbounded resolutions. For this we will adapt the famous Theorem 3.4 in [Spa88]. In the following we shall let \mathcal{B} be a class of complexes in \mathcal{E} which is stable under shifts, and we shall assume that for any bounded below complex X_\bullet there is a bounded below complex B_\bullet in \mathcal{B} and a quasi-isomorphism $B_\bullet \rightarrow X_\bullet$ which is an admissible epimorphism in each degree. We will call such a class a **bounded resolving class**. Let us recall some notions from Splanstein's paper.

Definition 2.52. Let \mathcal{B} be a class of complexes. A direct system $(P_\bullet^n)_{n \in E}$ in $\mathbf{Ch}(\mathcal{E})$ is a **\mathcal{B} -special direct system** if it satisfies the following conditions.

- (1) E is well-ordered.
- (2) If $n \in E$ has no predecessor then $P_\bullet^n = \lim_{\rightarrow_{m < n}} P_\bullet^m$.
- (3) If $n \in E$ has a predecessor $n-1$ then the natural chain map $P_\bullet^{n-1} \rightarrow P_\bullet^n$ is injective, its cokernel C_\bullet^n belongs to \mathcal{B} , and the short exact sequence

$$0 \rightarrow P_\bullet^{n-1} \rightarrow P_\bullet^n \rightarrow C_\bullet^n \rightarrow 0$$

is split exact in each degree.

We denote by $\lim_{\rightarrow} \mathcal{B}$ the class of complexes which are limits of \mathcal{B} -special direct systems.

Proposition 2.53. Let \mathcal{E} be an exact category which has kernels such that the functor $\lim_{\rightarrow n}$ is exact. Suppose that \mathcal{B} is a bounded resolving class. Then for any complex X_\bullet there exists a \mathcal{B} -special direct system $(P_\bullet^n)_{n \geq -1}$ and a direct system of chain maps $f^n : P_\bullet^n \rightarrow \tau_{\geq n} X_\bullet$ such that

- (1) f^n is a quasi-isomorphism for every $n \geq 0$.
- (2) f^n is an admissible epimorphism in each degree.

Proof. We construct the data $(P_\bullet^n)_{n \geq -1}$ and $(f^n)_{n \geq -1}$ by induction. For $n = -1$ we take $P_\bullet^{-1} = 0$ and so $f^{-1} = 0$. Let now $n \geq 1$, and suppose that $P_\bullet^{-1}, \dots, P_\bullet^{n-1}$ and f^{-1}, \dots, f^{n-1} have been constructed. Let $P_\bullet = P_\bullet^{n-1}$ and $Y_\bullet = \tau_{\geq n} X_\bullet$. Denote by f the composite $P_\bullet^{n-1} \rightarrow \tau_{\geq n-1} X_\bullet \rightarrow Y_\bullet$. By assumption we can find a quasi-isomorphism $g : Q_\bullet \rightarrow \text{cone}(f)[1]$ which is an admissible epimorphism in each degree, and $Q_\bullet[-1] \in \mathcal{B}$. Now we have a degree wise splitting $\text{cone}(f)[1] = P_\bullet \oplus Y_\bullet[1]$. We therefore get two maps $g' : Q_\bullet \rightarrow P_\bullet$ and $g'' : Q_\bullet \rightarrow Y_\bullet[1]$ which are admissible epimorphisms in each degree, and such that g' is a chain map. Let $h_n : \text{cone}(-g')_n = Q[1]_n \oplus P_n \rightarrow Y_n$ be defined by $h = g''[1] + f$. As in [Spa88], by direct calculation h is a chain map and $\text{cone}(h) = \text{cone}(g)[1]$. Since g is a quasi-isomorphism h is as well. Moreover the sequence

$$0 \rightarrow P_\bullet^{n-1} \rightarrow P_\bullet^n \rightarrow Q_\bullet[1] \rightarrow 0$$

is split exact in each degree. \square

Corollary 2.54. *Let \mathcal{E} be an exact category with kernels in which the direct limit functor $\lim_{\rightarrow \mathbb{N}}$ exists and is exact. Let \mathcal{B} be a bounded resolving class. Then any chain complex X_\bullet in \mathcal{E} admits a $\lim_{\rightarrow \mathcal{B}}$ resolution which is an admissible epimorphism in each degree.*

Proof. Fix a \mathcal{B} -special direct system $(P_\bullet^n)_{n \geq -1}$ and a direct system of chain maps $f^n : P_\bullet^n \rightarrow \tau_{\geq n} X_\bullet$ such that

- (1) f^n is a quasi-isomorphism for every $n \geq 0$.
- (2) f^n is an admissible epimorphism in each degree.

Let P_\bullet be the direct limit of the special direct system. For each n the composition $P_\bullet^n \rightarrow P_\bullet \rightarrow X_\bullet$ is an admissible epimorphism in degrees $> n$. Thus $P_\bullet \rightarrow X_\bullet$ is an admissible epimorphism in all degrees. \square

Now let \mathcal{P} be any class of objects in \mathcal{E} . Suppose that for each object X in \mathcal{E} there is an object P in \mathcal{P} together with an admissible epimorphism $P \twoheadrightarrow X$. By Lemma 2.50 the class $\mathbf{Ch}_+(\mathcal{P})$ of chain complexes with entries in \mathcal{P} is a bounded resolving class. Let us introduce the following notion.

Definition 2.55. *Let \mathcal{E} be an exact category such that for some ordinal λ , λ -indexed transfinite compositions of admissible monomorphisms exist. We say that a class of objects \mathcal{P} in \mathcal{E} is **closed under λ -indexed extensions** if for any continuous functor $X : \lambda \rightarrow \mathcal{E}$ functor such that for each $i < j$ in λ the map $X_i \rightarrow X_j$ is an admissible monic whose cokernel is in \mathcal{P} , then the limit X_λ is in \mathcal{P} .*

From the proof of Corollary 2.54 we then immediately have the following.

Corollary 2.56. *Let \mathcal{E} be an exact category with kernels in which the direct limit functor $\lim_{\rightarrow \mathbb{N}}$ exists and is exact. Let \mathcal{P} be a class of objects such that for each object X in \mathcal{E} there is an object P in \mathcal{P} together with an admissible epimorphism $P \twoheadrightarrow X$. Suppose further that \mathcal{P} is closed under \mathbb{N} -indexed extensions. Then for any complex X_\bullet in $\mathbf{Ch}(\mathcal{E})$ there is a complex P_\bullet in $\mathbf{Ch}(\mathcal{P})$ and an admissible epimorphism $P_\bullet \rightarrow X_\bullet$ which is a quasi-isomorphism. Moreover, X_\bullet is the limit of a $\mathbf{Ch}_+(\mathcal{P})$ -special direct system.*

We will also need the following acyclicity result, also proved in [Spa88] for abelian categories.

Proposition 2.57. *Let \mathcal{T} be a class of complexes in $\mathbf{Ch}(\mathcal{E})$. The class of all complexes $A_\bullet \in \mathbf{Ch}(\mathcal{E})$ such that $\mathbf{Hom}(A_\bullet, T_\bullet)$ is acyclic for every T_\bullet in \mathcal{T} is closed under special direct limits.*

Proof. It is clear from the definition of the contravariant functor $\mathbf{Hom}(-, T_\bullet)$ that it transforms colimits into limits. If $(P_\bullet^n)_{n \in E}$ is a \mathcal{B} -special direct system then $(\mathbf{Hom}(P_\bullet^n, T_\bullet))_{n \in E}$ is a \mathcal{B} -special inverse system of acyclic complexes of abelian groups, where we use the terminology of [Spa88]. Lemma 2.3 in [Spa88] says that the inverse limit of such a system is again acyclic. \square

2.5. Diagrams in Exact Categories. In this subsection we study exact structures on categories of diagrams in exact categories, in particular chain complexes in exact categories. Let \mathcal{I}, \mathcal{C} be categories. For each pair of morphisms $f, g \in \mathcal{I}$ such that $f \circ g$ is well-defined, let $\mathcal{M}_{(f,g)}$ be a full subcategory of the arrow category $\mathbf{Mor}(\mathcal{C})$. Write $\mathcal{M} = \{\mathcal{M}_{(f,g)} : f \circ g \text{ is well-defined}\}$. Denote by $[\mathcal{I}, \mathcal{C}]_{\mathcal{M}}$ the full subcategory of the functor category $[\mathcal{I}, \mathcal{C}]$ consisting of functors G satisfying $G(f \circ g) \in \mathcal{M}_{(f,g)}$ for each pair of morphisms

$f, g \in \mathcal{J}$ such that $f \circ g$ is well-defined. The proof of the following proposition is straight-forward diagram chasing.

Proposition 2.58. *Let I be a category and $F : I \rightarrow [\mathcal{J}, \mathcal{C}]_{\mathcal{M}}$ be a diagram. Denote its value on an object $i \in I$ by $F(-)_i$. Then a (co)limit of F exists if for each object $j \in \mathcal{J}$ the (co)limit of the functor $F(j) : I \rightarrow \mathcal{C}$ exists, and for each pair of composable morphisms f, g in \mathcal{J} , the (co)limit of the functor $F(f \circ g) : I \rightarrow \mathbf{Mor}(\mathcal{C})$ exists in $\mathbf{Mor}(\mathcal{C})$ and is contained in $\mathcal{M}_{(f,g)}$. Moreover, the (co)limit is then given on objects by $j \mapsto (\text{co})\lim_{i \in I} F(j)_i$ and on morphisms by $g \mapsto (\text{co})\lim_{i \in I} F(g)_i$*

Corollary 2.59. *Let $(\mathcal{E}, \mathcal{Q})$ be an exact category and \mathcal{J} a small category. Let \mathcal{M} be as above. Suppose that for each pair of composable morphisms $f, g \in \mathcal{J}$, $\mathcal{M}_{(f,g)}$ is an additive subcategory of $\mathbf{Mor}(\mathcal{E})$ which is closed under taking finite limits and finite colimits in $\mathbf{Mor}(\mathcal{E})$. Then $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ is an additive category, and the class of kernel-cokernel pairs $\mathcal{Q}_{\mathcal{J}, \mathcal{M}}$ on $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ defined by $(i : A \rightarrow B, p : B \rightarrow C) \in \mathcal{Q}_{\mathcal{J}, \mathcal{M}}$ if and only if for each $j \in \mathbf{Ob}(\mathcal{J})$, $(i(j) : A(j) \rightarrow B(j), p(j) : B(j) \rightarrow C(j)) \in \mathcal{Q}$ defines an exact structure on $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$. Moreover if \mathcal{E} is quasi-abelian or abelian then so is $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$*

Proof. Diagram categories in pre-additive categories are pre-additive in a natural way. Clearly any full subcategory of a pre-additive category is pre-additive. Thus to show that $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ is additive it remains to demonstrate that it has finite biproducts. This follows from the previous proposition.

Let us show that a morphism $p : B \rightarrow C$ is an admissible epic in $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ if and only if each $p(j)$ is an admissible epic in \mathcal{E} . It is clear that if p is an admissible epic then so is each $p(j)$. Now let p be a morphism such that for each $j \in \mathcal{J}$, $p(j) : B(j) \rightarrow C(j)$ is an admissible epic. By Proposition 2.58 the kernel of p exists. Let us denote it by $i : A \rightarrow B$. Moreover for each $j \in \mathcal{J}$, $i(j) : A(j) \rightarrow B(j)$ is a kernel of $p(j) : B(j) \rightarrow C(j)$ and hence is an admissible monic. Thus $p : B \rightarrow C$ is an admissible epic. Likewise if $i : A \rightarrow B$ is such that $i(j)$ is an admissible monic in \mathcal{Q} for all $j \in \mathcal{J}$, then i is an admissible monic in $\mathcal{Q}_{\mathcal{J}, \mathcal{M}}$.

It is clear now the class of admissible epics and the class of admissible monics contain isomorphisms and are closed under composition. By Proposition 2.58, pushouts are defined object wise. It then follows from the previous remarks that pushouts of admissible monics are admissible monics. Likewise pullbacks of admissible epics are admissible epics.

For the last statement we use the criterion of Remark 2.81 (see later). Note that the condition of a morphism in $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ being weakly admissible is an object wise one. In particular if \mathcal{E} is quasi-abelian then every morphism in \mathcal{E} is weakly admissible, then so is every morphism in $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$. Thus $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ is also quasi-abelian. Noting that the property of a morphism being admissible is also an object-wise one, we see similarly that $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ is abelian whenever \mathcal{E} is. \square

Our main example is the category of chain complexes in an exact category.

Example 2.60. *Regard \mathbb{Z} as a poset, and consider the category \mathbb{Z}^{op} . Denote the morphism $n \rightarrow n-1$ in \mathbb{Z}^{op} by $[n]$. Let \mathcal{E} be a pre-additive category, and for each integer n , let $\mathcal{M}_{[n], [n-1]}$ be the full subcategory of $\mathbf{Mor}(\mathcal{E})$ consisting of the zero morphisms. Note that this uniquely determines the categories $\mathcal{M}_{f,g}$ for any pair of composable morphisms f, g in \mathcal{J} . Then there is an isomorphism of categories*

$$\mathbf{Ch}(\mathcal{E}) \cong [\mathbb{Z}^{op}, \mathcal{E}]_{\mathcal{M}}$$

The admissible monomorphisms (resp. epimorphisms) are the morphisms of chain complexes which are admissible monomorphisms (resp. epimorphisms) in each degree. Note also that for $$ in $\{\geq 0, \leq 0, +, -, b\}$, the categories $\mathbf{Ch}_*(\mathcal{E})$ are full extension-closed subcategories of $\mathbf{Ch}(\mathcal{E})$. Thus they inherit exact structures. Again the admissible monomorphisms (resp. epimorphisms) are the morphisms of chain complexes which are admissible monomorphisms (resp. epimorphisms) in each degree.*

It is now easy to construct a left abelianization of $\mathbf{Ch}_*(\mathcal{E})$, given a left abelianization $I : \mathcal{E} \rightarrow \mathcal{A}$ of \mathcal{E} .

Proposition 2.61. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a fully faithful exact functor which reflects exactness and whose essential image is closed under extensions. Then for $*$ in $\{\geq 0, \leq 0, +, -, b, \emptyset\}$ the induced functor*

$$\mathbf{Ch}_*(F) : \mathbf{Ch}_*(\mathcal{A}) \rightarrow \mathbf{Ch}_*(\mathcal{B})$$

is a fully faithful exact functor which reflects exactness and whose essential image is closed under extensions.

Proof. Since exactness of chain complexes is defined level wise, $\mathbf{Ch}_*(F)$ is clearly exact and reflects exactness. It is clearly faithful. Let us check that it is full. Let (X_\bullet, d_\bullet) and $(Y_\bullet, \delta_\bullet)$ be chain complexes in \mathcal{A} . Let $f_\bullet : F(X_\bullet) \rightarrow F(Y_\bullet)$ be a chain map. For each n there is some $g_n : X_n \rightarrow Y_n$ with $f_n = F(g_n)$. Moreover

$$F(g_n \circ d_{n+1}) = F(g_n) \circ F(d_{n+1}) = f_n \circ F(d_{n+1}) = F(\delta_{n+1}) \circ f_{n+1} = F(\delta_{n+1} \circ g_{n+1})$$

Since F is faithful, $g_n \circ d_{n+1} = \delta_{n+1} \circ g_{n+1}$. It remains to show that the essential image of $\mathbf{Ch}_*(F)$ is closed under extensions. So suppose we have an exact sequence of chain complexes.

$$0 \longrightarrow F(X_\bullet, d_\bullet) \xrightarrow{f_\bullet} (Q_\bullet, \gamma_\bullet) \xrightarrow{g_\bullet} F(Y_\bullet, \delta_\bullet) \longrightarrow 0$$

For each n pick an object $P_n \in \mathcal{A}$ and an isomorphism $p_n : Q_n \xrightarrow{\sim} F(P_n)$. Let $\gamma'_n = p_{n-1} \circ \gamma_n \circ p_n^{-1} : F(P_n) \rightarrow F(P_{n-1})$. Then $(P_\bullet, \gamma'_\bullet)$ is a chain complex. Moreover by construction we have an isomorphism $p_\bullet : Q_\bullet \rightarrow F(P_\bullet)$ whose n th component is p_n . \square

Corollary 2.62. *Let $I : \mathcal{E} \rightarrow \mathcal{A}(\mathcal{E})$ is a left abelianization of \mathcal{E} . Then $\mathbf{Ch}_*(I) : \mathbf{Ch}_*(\mathcal{E}) \rightarrow \mathbf{Ch}_*(\mathcal{A}(\mathcal{E}))$ is a left abelianization of $\mathbf{Ch}_*(\mathcal{E})$.*

Proof. By the previous proposition, it remains to check that $\mathbf{Ch}(I)$ preserves kernels, and $\mathbf{Ch}(I)(f_\bullet)$ is an admissible epimorphism if and only if f_\bullet is. However this is clear since everything is computed degree-wise. \square

2.5.1. A Useful Example: The Degree-wise Split structure. Let \mathcal{E} be an additive category, and endow it with the split exact structure. The induced exact structure on $\mathbf{Ch}(\mathcal{E})$ is called the **degree-wise split** exact structure, and we denote the ext functors in this structure by Ext_{dw}^n . We conclude this section with a brief discussion of the relation between extensions in the degree-wise split exact structure and the $\mathbf{Ch}(\mathbf{Ab})$ -enriched structure on $\mathbf{Ch}(\mathcal{E})$. This is also done in a model theoretic context for modules over a ring in [Gil11].

Proposition 2.63. *A sequence of chain complexes $0 \longrightarrow X_\bullet \xrightarrow{p_\bullet} Z_\bullet \xrightarrow{q_\bullet} Y_\bullet \longrightarrow 0$ is split exact in each degree if and only if it is isomorphic to a complex of the form*

$$0 \rightarrow X_\bullet \rightarrow \text{cone}(f_\bullet) \rightarrow Y_\bullet \rightarrow 0$$

for some morphism of complexes $f_\bullet : Y_\bullet[1] \rightarrow X_\bullet$.

Proof. The sequence

$$0 \rightarrow X_\bullet \rightarrow \text{cone}(f_\bullet) \rightarrow Y_\bullet \rightarrow 0$$

is clearly split exact in each degree, so any complex isomorphic to it is split exact in each degree as well.

Suppose

$$0 \longrightarrow X_\bullet \xrightarrow{p_\bullet} Z_\bullet \xrightarrow{q_\bullet} Y_\bullet \longrightarrow 0$$

is split exact in each degree. Let $\alpha_n : Z_n \rightarrow X_n$ be such that $\alpha_n \circ p_n = \text{id}_{X_n}$ and $\beta_n : Y_n \rightarrow Z_n$ be a map such that $q_n \circ \beta_n = \text{id}_{Y_n}$. We may assume also that $\alpha_n \circ \beta_n = 0$. Define $f_\bullet : Y_\bullet[1] \rightarrow X_\bullet$ by $f_n = \alpha_n \circ d_{n+1}^Z \circ \beta_{n+1}$. This is easily seen to be a map of chain complexes. Let $\alpha_n : Z_n \rightarrow X_n \oplus Y_n$ denote the isomorphism induced by the degree-wise splitting. A straight-forward computation shows that this gives a map of chain complexes $\alpha_\bullet : Z_\bullet \rightarrow \text{cone}(f_\bullet)$. Thus we get an isomorphism of exact sequences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_\bullet & \xrightarrow{p_\bullet} & Z_\bullet & \xrightarrow{q_\bullet} & Y_\bullet \longrightarrow 0 \\ & & \parallel & & \downarrow \alpha_\bullet & & \parallel \\ 0 & \longrightarrow & X_\bullet & \longrightarrow & \text{cone}(f_\bullet) & \longrightarrow & Y_\bullet \longrightarrow 0 \end{array}$$

\square

Proposition 2.64. *A map of chain complexes $f_\bullet : X_\bullet \rightarrow Y_\bullet$ is homotopic to 0 if and only if the sequence*

$$0 \rightarrow Y_\bullet \rightarrow \text{cone}(f_\bullet) \rightarrow X_\bullet[-1] \rightarrow 0$$

is split exact.

Proof. Suppose that f_\bullet is homotopic to 0. Let $\{D_n : X_n \rightarrow Y_{n+1}\}$ be a homotopy. We then get a map $\alpha_n = (\text{id}_{X_{n-1}}, D_{n-1}) : X_{n-1} \rightarrow \text{cone}(f)_n$. It is straight-forward to check that this gives a chain map $\alpha_\bullet : X_\bullet[-1] \rightarrow \text{cone}(f_\bullet)$. Moreover it obviously gives a splitting of $\text{cone}(f_\bullet) \rightarrow X_\bullet[-1]$.

Conversely suppose the sequence is split exact. Let $\alpha_\bullet : X_\bullet[-1] \rightarrow \text{cone}(f_\bullet)$ be a splitting of the map $\text{cone}(f_\bullet) \rightarrow X_\bullet[-1]$. It is an easy computation to check that the collection of compositions $\{D_{n-1} : X_{n-1} \rightarrow \text{cone}(f_\bullet)_n \rightarrow Y_n\}$ is a homotopy between f and 0. \square

We recover the following standard result

Corollary 2.65. *For chain complexes X_\bullet, Y_\bullet in an additive category \mathcal{E} , we have*

$$\text{Ext}_{dw}^1(X, Y[n-1]) \cong H_n \mathbf{Hom}(X_\bullet, Y_\bullet) = \text{Hom}_{\mathbf{Ch}(\mathcal{E})}(X, Y[n]) / \sim$$

where \sim is chain homotopy.

Proof. By direct computation, one finds that $f \in \prod_i \text{Hom}(X_i, Y_{i+n})$ defines a chain map $f_\bullet : X_\bullet \rightarrow Y_\bullet[n]$ if and only if $f \in \text{Ker}(d_n)$. Similarly, f_\bullet is then null-homotopic if and only if it is in $\text{Im}(d_{n+1})$. This gives the isomorphism

$$H_n \mathbf{Hom}(X_\bullet, Y_\bullet) = \text{Hom}_{\mathbf{Ch}(\mathcal{E})}(X, Y[n]) / \sim$$

The isomorphism $\text{Ext}_{dw}^1(X, Y[n-1]) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{E})}(X, Y[n]) / \sim$ follows from Proposition 2.63 and Proposition 2.64. \square

2.6. Monoidal Exact Categories and Monads in Exact Categories. We conclude this section with a brief note on monoidal exact categories, and exact structures on categories of modules over monoids. More generally we put an exact structure on the category of algebras for an additive monad which is compatible in a precise sense with the exact structure on the underlying category. First we need a general definition.

Definition 2.66. *A covariant functor $F : \mathcal{E} \rightarrow \mathcal{F}$ between exact categories is said to be **right exact** if for any short exact sequence*

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

in \mathcal{E} , the sequence

$$F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow 0$$

is admissibly coacyclic in \mathcal{F} .

*A contravariant functor $F : \mathcal{E} \rightarrow \mathcal{F}$ between exact categories is said to be **right exact** if for any short exact sequence*

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

in \mathcal{E} , the sequence

$$F(Z) \rightarrow F(Y) \rightarrow F(X) \rightarrow 0$$

is admissibly coacyclic in \mathcal{F} .

Dually one defines left exactness.

Definition 2.67. *Let \mathcal{E} be an exact category. A symmetric monoidal structure with additive tensor functor \otimes is said to be **compatible** if for any object X of \mathcal{E} the functor $X \otimes (-)$ preserves all colimits which exist and is right exact. A **monoidal exact category** is a symmetric monoidal category $(\mathcal{E}, \otimes, k)$ where \mathcal{E} is an exact category and the monoidal structure is compatible.*

Definition 2.68. Let \mathcal{E} be an exact category. A closed symmetric monoidal structure with additive tensor functor \otimes and additive internal hom $\underline{\text{Hom}}$ is said to be **compatible** with \mathcal{E} if for each object X of \mathcal{E} , the functor $X \otimes (-)$, is right exact and the functors $\underline{\text{Hom}}(A, -)$ and $\underline{\text{Hom}}(-, A)$ are left exact. A **closed monoidal exact category** is a closed monoidal category $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ where \mathcal{E} is an exact category and the closed monoidal structure is compatible.

Note that if $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ is a closed monoidal exact category, then $(\mathcal{E}, \otimes, k)$ is automatically a monoidal exact category. Indeed for each object X , $X \otimes (-)$ is a left adjoint so it preserves colimits.

Definition 2.69. Let $(\mathcal{E}, \otimes, k)$ be an exact category equipped with a (not necessarily compatible) symmetric monoidal structure where the tensor functor is additive. An object F of \mathcal{E} is said to be **(strongly) flat** if the functor $F \otimes (-)$ is (strongly) exact.

In the familiar category of R -modules over some ring R with the usual monoidal structure, projectives are always flat. Moreover the tensor product of two projective R -modules is again projective. This is not always guaranteed for an arbitrary monoidal exact category. However it is a useful property to have, in particular when dealing with the projective model structure later. We therefore make a definition.

Definition 2.70. A monoidal exact category in which projective objects are flat and $P \otimes P'$ is projective whenever both P and P' are is said to be **projectively monoidal**. A projectively monoidal exact category is said to be **strongly projectively monoidal** if projectives are strongly flat.

In closed exact categories we have the following observation.

Observation 2.71. Let $(\mathcal{E}, \otimes, k, \underline{\text{Hom}})$ be a closed monoidal exact category with enough projectives such that the underlying monoidal category is projectively monoidal. Then for any projective P , the functor $\underline{\text{Hom}}(P, -) : \mathcal{E} \rightarrow \mathcal{E}$ is exact. The proof follows immediately from the adjunction between \otimes and $\underline{\text{Hom}}$. It is shown in the quasi-abelian case in [Sch99], for example, and the proof works identically in the exact case.

Now let R be a unital associative monoid internal to a monoidal exact category $(\mathcal{E}, \otimes, k)$. It turns out that there is an exact structure on the additive category $R - \mathbf{Mod}$ where a null sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

in $R - \mathbf{Mod}$ is exact if and only if it is a short exact sequence when regarded as a null-sequence in \mathcal{E} . This follows from a more general result about compatible monads in exact categories.

Definition 2.72. An additive monad T on an exact category \mathcal{E} is said to be **compatible** if it preserves all colimits and is a right exact functor.

Proposition 2.73. Let \mathcal{E} be an exact category and let $T : \mathcal{E} \rightarrow \mathcal{E}$ be a compatible monad. There is an exact structure on \mathcal{E}^T where a null sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

in \mathcal{E}^T is exact if and only if it is a short exact sequence when regarded as a null-sequence in \mathcal{E} . We call this exact structure the **induced exact structure**.

Proof. This follows from the general fact that if T is a cocontinuous monad on any category \mathcal{E} then the forgetful functor $\mathcal{E}^T \rightarrow \mathcal{E}$ creates limits and colimits and reflects isomorphisms. For a proof see [Bor94] Proposition 4.3.1 and Proposition 4.3.2. \square

This exact structure inherits a lot from the exact structure on \mathcal{E} . In fact we have the following lemma.

Lemma 2.74. Let $|-| : \mathcal{D} \rightarrow \mathcal{E}$ be a functor between exact categories which reflects exactness and creates both kernels and cokernels. Then $|-|$ reflects admissible monomorphisms, admissible epimorphisms, weakly admissible morphisms, admissible morphisms, and admissibly acyclic sequences.

Proof. Let $f : X \rightarrow Y$ be a morphism in \mathcal{D} . Suppose that $|f|$ is an admissible monomorphism. Then there is an exact sequence

$$0 \rightarrow |X| \rightarrow |Y| \rightarrow \text{Coker}(|f|) \rightarrow 0$$

Since $|-|$ creates cokernels and reflects exactness

$$0 \rightarrow X \rightarrow Y \rightarrow \text{Coker}(f) \rightarrow 0$$

is an exact sequence in \mathcal{D} . Thus f is an admissible monomorphism. That $|-|$ reflects admissible epimorphisms is proved similarly. Note in particular that this means $|-|$ reflects isomorphisms.

Suppose now that $|f| : |X| \rightarrow |Y|$ is weakly admissible. Then there is a decomposition

$$\begin{array}{ccccc} & & |X| & \xrightarrow{|f|} & |Y| \\ & \nearrow & & & \searrow \\ \text{Ker}(|\hat{f}|) & & & & \text{Coker}(|f|) \\ & \searrow & & \nearrow & \\ & & \text{Coim}(|f|) & \xrightarrow{|\hat{f}|} & \text{Im}(|f|) \end{array}$$

Since $|-|$ reflects exactness and creates both kernels and cokernels there is a decomposition in \mathcal{D}

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y \\ & \nearrow & & & \searrow \\ \text{Ker}(f) & & & & \text{Coker}(f) \\ & \searrow & & \nearrow & \\ & & \text{Coim}(f) & \xrightarrow{\hat{f}} & \text{Im}(f) \end{array}$$

Thus f is weakly admissible. If in addition $|f|$ is admissible then $|\hat{f}|$ is an isomorphism. Since $|-|$ reflects isomorphisms \hat{f} is an isomorphism, so f is admissible.

Finally suppose

$$|X| \xrightarrow{|f|} |Y| \xrightarrow{|g|} |Z|$$

is admissibly acyclic. Then $|f|$ is admissible, $|g|$ has a kernel and the map $\text{Im}(|f|) \rightarrow \text{Ker}(|g|)$ is an isomorphism. By the above f is admissible. Since $|-|$ creates kernels and cokernels and also reflects isomorphisms $\text{Im}(f) \rightarrow \text{Ker}(g)$ is also an isomorphism. \square

As a consequence of this and Remark 2.81 later, if \mathcal{E} is (quasi)-abelian, then so is \mathcal{E}^T . In particular categories of modules for monoid objects in monoidal (quasi)-abelian categories are themselves (quasi)-abelian.

Before concluding this discussion of monoidal exact categories, let us briefly mention induced monoidal structures on chain complexes. So, let $(\mathcal{E}, \otimes, k)$ be a monoidal exact category. Recall from Section 1 there is an induced additive monoidal structure $(\mathbf{Ch}_*(\mathcal{E}), \otimes, S^0(k))$ on $\mathbf{Ch}_*(\mathcal{E})$ for $* \in \{\geq 0, \leq 0, +, -, b, \emptyset\}$. Since colimits of chain complexes are computed degreewise, finite direct sums are strongly exact, and a null-sequence of chain complexes is admissibly coacyclic if and only if it is so in each degree, it is clear that this monoidal structure is compatible, so that $(\mathbf{Ch}_*(\mathcal{E}), \otimes, S^0(k))$ is a monoidal exact category for $* \in \{\geq 0, \leq 0, +, -, b\}$.

Now suppose $(\mathcal{E}, \otimes, S^0(k), \underline{\text{Hom}})$ is a closed monoidal exact category. Then

$$(\mathbf{Ch}_b(\mathcal{E}), \otimes, S^0(k), \underline{\text{Hom}})$$

is a closed monoidal exact category. Note that the closed symmetric monoidal category

$$(\mathbf{Ch}(\mathcal{E}), \otimes, S^0(k), \underline{\text{Hom}})$$

need not be a closed monoidal exact category since infinite direct sums/ products need not be admissibly coexact/ admissibly exact. When we deal with unbounded complexes later we shall assume this to be the case. We shall see shortly that this is guaranteed for a closed monoidal structure on a quasi-abelian category.

2.7. Quasi-Abelian Categories. Let us apply what we have seen so far to the particular case of quasi-abelian categories. The theory of quasi-abelian categories is developed significantly in [Sch99] which is our main reference here. Applications to categories of topological vector spaces can be found in [Pro00].

2.7.1. Strict Morphisms. First we show that Definition 2.5 is equivalent to the one given in [Sch99]. Recall that in a finitely bicomplete additive category, any morphism $f : E \rightarrow F$ gives rise to a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow & & \uparrow \\ \text{Coim}(f) & \longrightarrow & \text{Im} f \end{array}$$

In any abelian category the map $\text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism. However this is not true in general. For example, consider the standard example of the category \mathbf{Fr} of Fréchet spaces. Then $\text{Coim}(f) = E/f^{-1}(0)$, $\text{Im}(f) = \overline{f(E)}$ and the natural map $E/f^{-1}(0) \rightarrow \overline{f(E)}$ is the obvious one. By the Open Mapping Theorem $\text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism if and only if f has closed range, which is not always the case.

Definition 2.75. Let \mathcal{E} be an additive category with all kernels and cokernels. A morphism $f : E \rightarrow F$ in \mathcal{E} is said to be **strict** if $\text{Coim}(f) \rightarrow \text{Im}(f)$ is an isomorphism.

Proposition 2.76. Let \mathcal{E} be a finitely bicomplete additive category.

- (1) A monic is strict if and only if it is the kernel of some morphism. In this case it is the kernel of its cokernel.
- (2) An epic is strict if and only if it is the cokernel of some morphism. In this case it is the cokernel of its kernel.

Proof. (1) Let $f : E \rightarrow F$ and write $i_f : \text{Ker}(f) \rightarrow E$ for the canonical map. Let us show that i_f is strict. First note that for any monic $A \rightarrow B$, the coimage is $\text{id} : A \rightarrow A$. Let us compute the image of i_f . It is given by

$$\text{Ker}(\text{Coker}(\text{Ker}(f) \rightarrow E) \rightarrow E)$$

By some abstract nonsense this is just $\text{Ker}(f) \rightarrow E$.

Conversely suppose $m : X \rightarrow E$ is a strict monic. Then the maps $E \rightarrow \text{Coim}(m) \rightarrow \text{Im}(m)$ are all isomorphisms, i.e. we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{m} & E \\ \downarrow \sim & & \uparrow \\ \text{Coim}(m) & \xrightarrow{\sim} & \text{Im}(m) \end{array}$$

Since $\text{Im}(m) \rightarrow E$ is a kernel of $\text{Coker}(m)$, so is $m : X \rightarrow E$.

- (2) This is dual to the first part.

□

Proposition 2.77. The class of strict epics (resp. monics) of \mathcal{E} is stable by composition.

Proof. See [Sch99] Proposition 1.1.7.

□

Corollary 2.78. A finitely bicomplete additive category \mathcal{E} is quasi-abelian if and only if the following two conditions hold:

(1) If

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f'} & Y \end{array}$$

is a push out diagram, and f is a strict monic, then f' is as well.

(2) If

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback diagram, and f is a strict epic, then f' is as well.

Let us now describe the admissible morphisms in the quasi-abelian exact structure.

Proposition 2.79. *Let \mathcal{E} be a finitely bicomplete additive category. A morphism $f : E \rightarrow F$ in \mathcal{E} is strict if and only if it can be written as $f = i \circ p$ where $p : E \rightarrow I$ is a strict epic and $i : I \rightarrow F$ is a strict monic.*

Proof. Suppose f admits a decomposition $f = i \circ p$ as in the statement. Then $\text{Ker}(f) = \text{Ker}(p)$. So $\text{Coim}(f) = \text{Coim}(p)$. Since p is strict $\text{Coim}(p) \cong \text{Im}(p)$. Since p is an epic, $\text{Im}(p) = I$. Similarly $\text{Im}(f) = \text{Im}(i) = I$.

Conversely suppose f is a strict morphism. Now $E \rightarrow \text{Coim}(f)$ is a strict epic, and $\text{Im}(f) \rightarrow F$ is a strict monic. But since f is strict, $\text{Coim}(f) \cong \text{Im}(f)$, so this gives the decomposition of f . \square

We immediately get:

Corollary 2.80. *A morphism in a quasi-abelian category is admissible in the quasi-abelian exact structure if and only if it is strict.*

Remark 2.81. *An exact structure on a finitely bicomplete additive category coincides with the quasi-abelian structure if and only if every morphism is weakly admissible. Then as a consequence of Proposition 2.76, a finitely bicomplete additive category is abelian if and only if every morphism is admissible.*

2.7.2. The Left Heart. Homology in quasi-abelian is significantly easier than in more general exact categories. For example, there is an even stronger abelian embedding.

Theorem 2.82. *Let \mathcal{E} be a quasi-abelian category. There exists a left abelianization $I : \mathcal{E} \rightarrow \mathcal{LH}(\mathcal{E})$ of \mathcal{E} such that I has a left adjoint $C : \mathcal{LH}(\mathcal{E}) \rightarrow \mathcal{E}$ with $C \circ I \cong \text{id}_{\mathcal{E}}$, i.e. \mathcal{E} is a reflective subcategory of $\mathcal{LH}(\mathcal{E})$. Moreover the induced functor on derived categories*

$$D(I) : D(\mathcal{E}) \rightarrow D(\mathcal{LH}(\mathcal{E}))$$

is an equivalence.

Proof. See [Sch99] Proposition 1.1.26, Corollary 1.2.27, Proposition 1.2.28, and Proposition 1.2.31. \square

$\mathcal{LH}(\mathcal{E})$ is called the **left heart** of \mathcal{E} . The embedding of \mathcal{E} into its left heart also behaves extremely well with respect to projectives, namely:

Proposition 2.83. (1) *An object P of \mathcal{E} is projective if and only if $I(P)$ is projective in $\mathcal{LH}(\mathcal{E})$.*

(2) *\mathcal{E} has enough projectives if and only if $\mathcal{LH}(\mathcal{E})$ has enough projectives. In this case an object of $\mathcal{LH}(\mathcal{E})$ is projective if and only if it is isomorphic to $I(P)$ where P is projective in \mathcal{E} .*

Proof. See [Sch99] Proposition 1.3.24. \square

Moreover left abelianizations of quasi-abelian categories allow us to test acyclicity of any unbounded complex. Indeed as a consequence of Remark 2.81 and Corollary 2.30 we get.

Corollary 2.84. *Let $I : \mathcal{E} \rightarrow \mathcal{A}$ be a left abelianisation of \mathcal{E} where \mathcal{E} is a quasi-abelian category. Then a complex X_\bullet in \mathcal{E} is acyclic if and only if $I(X_\bullet)$ is acyclic. In particular a map of complexes $f : X \rightarrow Y$ is a quasi-isomorphism if and only if $I(f)$ is.*

2.7.3. Monoidal Quasi-Abelian Categories. Let us briefly discuss (strongly) projectively monoidal quasi-abelian categories, i.e. a (strongly) projectively monoidal exact category in which the underlying exact category is quasi-abelian. We first make the following observation.

Observation 2.85. *An additive functor $F : \mathcal{E} \rightarrow \mathcal{F}$ between quasi-abelian categories is right exact if and only if it preserves cokernels of strict morphisms.*

This implies that if $(\mathcal{E}, \otimes, k)$ is a monoidal category with \mathcal{E} quasi-abelian and \otimes additive, then it is a monoidal quasi-abelian category if and only if $X \otimes (-)$ preserves colimits for each object X of \mathcal{E} . In particular if $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ is a closed monoidal category with \mathcal{E} quasi-abelian and $\otimes, \underline{\text{Hom}}$ additive functors, then $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ is in fact a closed monoidal quasi-abelian category.

Proposition 2.86. *Let $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ be a bicomplete closed monoidal quasi-abelian category which is also projectively monoidal. Then there is a monoidal structure $\tilde{\otimes}, \tilde{\underline{\text{Hom}}}$ on $\mathcal{LH}(\mathcal{E})$ such that $(\mathcal{LH}(\mathcal{E}), \tilde{\otimes}, \tilde{\underline{\text{Hom}}}, I(k))$ is a closed monoidal abelian category. Moreover $I : \mathcal{E} \rightarrow \mathcal{LH}(\mathcal{E})$ is a lax monoidal functor. If $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ is strongly projectively monoidal then $(\mathcal{LH}(\mathcal{E}), \tilde{\otimes}, \tilde{\underline{\text{Hom}}}, I(k))$ is projectively monoidal.*

Proof. See [Sch99] Proposition 1.5.3 and Corollary 1.5.4. □

3. GENERATORS IN EXACT CATEGORIES

In this section we will introduce suitable notions of a generating set in an exact category. This will come into play later when we discuss cofibrant generation of model structures, where some compactness assumptions are required. For our definition of generating set we will generalise an equivalent characterisation ([Sch99] Proposition 2.1.7) of Schneiders' notion of a **strictly generating set** in a quasi-abelian category, [Sch99] Definition 2.1.5. If \mathcal{G} is a collection of objects in an exact category we denote by $\bigoplus \mathcal{G}$ the collection of all small coproducts of objects in \mathcal{G} .

Definition 3.1. *A collection of objects \mathcal{G} in an exact category \mathcal{E} is said to be an **admissible generating set** if for each object E of \mathcal{E} there is an object Q of $\bigoplus \mathcal{G}$ and an admissible epimorphism $Q \twoheadrightarrow E$. An admissible generating set \mathcal{G} is said to be a **projective generating set** if all objects in \mathcal{G} are projective.*

The next two results are adaptations of the proof of [Sch99] Proposition 1.3.23 to the exact case.

Proposition 3.2. *Let \mathcal{G} be an admissible generating set. Suppose $f : E \rightarrow F$ is a morphism such that for each G in \mathcal{G} then map $\text{Hom}(G, E) \rightarrow \text{Hom}(G, F)$ is an epimorphism. Then f is an admissible epimorphism.*

Proof. Pick an admissible epimorphism $\epsilon : P \rightarrow F$ where $P \in \bigoplus \mathcal{G}$. By assumption there is a morphism $\epsilon' : P \rightarrow E$ such that $\epsilon = f \circ \epsilon'$. By Proposition 2.9 f is then an admissible epimorphism. □

Proposition 3.3. *Let \mathcal{G} be a generating set in an exact category \mathcal{E} . A complex*

$$0 \longrightarrow E \xrightarrow{e'} E \xrightarrow{e''} E''$$

with e'' weakly left admissible is admissibly acyclic if and only if for each $G \in \mathcal{G}$ the sequence

$$0 \longrightarrow \text{Hom}(G, E') \longrightarrow \text{Hom}(G, E) \longrightarrow \text{Hom}(G, E'')$$

*is acyclic in **Ab**. If in addition the objects of \mathcal{G} are projective, then a sequence*

$$E \xrightarrow{e'} E \xrightarrow{e''} E''$$

with e'' weakly left admissible is admissibly acyclic if and only if for each $P \in \mathcal{P}$ the sequence

$$\mathrm{Hom}(P, E') \longrightarrow \mathrm{Hom}(P, E) \longrightarrow \mathrm{Hom}(P, E'')$$

is acyclic in **Ab**.

Proof. Suppose that for each $G \in \mathcal{G}$ the sequence

$$0 \longrightarrow \mathrm{Hom}(G, E') \longrightarrow \mathrm{Hom}(G, E) \longrightarrow \mathrm{Hom}(G, E'')$$

is acyclic in **Ab**. Since e'' is weakly left admissible it is sufficient to show that e' is a kernel of e'' . Then e' is automatically an admissible monic. To show this one can follow the proof in [Sch99]. At one point in that proof the existence of a resolution of X by objects of $\oplus \mathcal{G}$ is used. Here instead we may use Lemma 2.50

Finally let us consider the assertion about projective generators. Proposition 2.25 implies that

$$\mathrm{Hom}(P, E') \longrightarrow \mathrm{Hom}(P, E) \longrightarrow \mathrm{Hom}(P, E'')$$

is acyclic. For the converse first consider the sequence

$$0 \longrightarrow \mathrm{Ker}(e'') \longrightarrow E \xrightarrow{e''} E''$$

Since $\mathrm{Hom}(P, -)$ preserves kernels, Proposition 3.2 implies that

$$E \xrightarrow{e'} E \xrightarrow{e''} E''$$

is admissibly acyclic. □

In particular if \mathcal{E} is quasi-abelian, then every morphism is weakly admissible, so in this case one has that a sequence

$$E \xrightarrow{e'} E \xrightarrow{e''} E''$$

is admissibly acyclic if and only if for each $P \in \mathcal{P}$ the sequence

$$\mathrm{Hom}(P, E') \longrightarrow \mathrm{Hom}(P, E) \longrightarrow \mathrm{Hom}(P, E'')$$

is acyclic in **Ab**. For general exact categories we still have the following result.

Corollary 3.4. *Let \mathcal{G} be a projective generating set in an exact category \mathcal{E} . Let X_\bullet be a complex. Suppose that X_\bullet is good. Then X_\bullet is acyclic if and only if $\mathrm{Hom}(G, X_\bullet)$ is acyclic for each $G \in \mathcal{G}$.*

Proof. Since each $G \in \mathcal{G}$ is projective the functors $\mathrm{Hom}(G, -)$ preserve acyclic complexes.

Conversely suppose $\mathrm{Hom}(G, X_\bullet)$ is acyclic for each $G \in \mathcal{G}$, and d_n^X has a kernel $Z_n X$. By assumption $\mathrm{Hom}(G, d'_{n+1}) : \mathrm{Hom}(G, X_{n+1}) \rightarrow Z_n \mathrm{Hom}(G, X) = \mathrm{Hom}(G, Z_n X)$ is an epimorphism for each n . Thus $d'_{n+1} : X_{n+1} \rightarrow Z_n X$ is an admissible epimorphism. Now apply Proposition 2.29. □

3.1. Elementary Exact Categories. It is convenient to have generators satisfying some smallness conditions. We will use the same terminology as Schneiders does in [Sch99] for quasi-abelian categories.

Definition 3.5. *Let \mathcal{E} be an additive category. An object E of \mathcal{E} is said to be*

(1) **small** if the natural map

$$\mathrm{Hom}\left(E, \bigoplus_{i \in I} F_i\right) \rightarrow \bigoplus_{i \in I} \mathrm{Hom}(E, F_i)$$

is an isomorphism for any small family $(F_i)_{i \in I}$ of \mathcal{E} whose direct sum exists.

(2) **tiny** if the natural map

$$\mathrm{Hom}(E, \lim_{\rightarrow i \in \mathcal{I}} F_i) \rightarrow \lim_{\rightarrow i \in \mathcal{I}} \mathrm{Hom}(E, F_i)$$

is an isomorphism for any filtering inductive system $E : \mathcal{I} \rightarrow \mathcal{E}$ whose direct limits exists.

Definition 3.6. *An exact category \mathcal{E} is said to be*

- (1) **projectively generated** if it has a projective generating set
- (2) **quasi-elementary** if it has a projective generating set consisting of small objects.
- (3) **elementary** if it has a projective generating set consisting of tiny objects.

Proposition 3.7. *A cocomplete quasi-abelian category is (quasi)-elementary if and only if its left heart is (quasi)-elementary.*

Proof. See [Sch99] Proposition 2.1.12. □

The following proposition is immediate from Proposition 3.3 and Corollary 3.4 but it has a useful consequence.

Proposition 3.8. *Let \mathcal{E} be a cocomplete elementary (resp. quasi-elementary) exact category. Then filtering inductive limits (resp. direct sums) in \mathcal{E} are exact. If in addition \mathcal{E} is quasi-abelian elementary (resp. quasi-elementary), then filtering inductive limits (resp. direct sums) are admissibly exact.*

Proposition 3.9. *Let \mathcal{E} be a cocomplete exact category in which the direct limit functor is exact. Then transfinite compositions of admissible monics are admissible monics.*

Proof. Since finite compositions of admissible monics are admissible, the successor case is clear. For the limit case let Λ be a limit ordinal, and consider the commutative diagram

$$\begin{array}{ccccccc}
 E_0 & \longrightarrow & E_0 & \longrightarrow & E_0 & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 E_0 & \xrightarrow{c_\lambda} & E_\lambda & \longrightarrow & E_{\lambda'} & \longrightarrow & \dots \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Coker}(c_\lambda) & \longrightarrow & \text{Coker}(c_{\lambda'}) & \longrightarrow & \dots
 \end{array}$$

with short exact columns. Taking the direct limit over Λ , we get a short exact sequence

$$0 \rightarrow E_0 \rightarrow E \rightarrow C \rightarrow 0$$

In particular $E_0 \rightarrow E$ is admissible. □

3.2. Generators in Categories of Chain Complexes. Our goal now is to show that if \mathcal{E} is an elementary exact category then so is $\mathbf{Ch}_*(\mathcal{E})$, for $*$ $\in \{+, \leq 0, -, b, \geq 0, \emptyset\}$. Much of this is based on the following technical result.

Lemma 3.10. *Let \mathcal{E} be a weakly idempotent complete exact category. For any object $C \in \mathcal{E}$ and $X, Y \in \mathbf{Ch}(\mathcal{E})$ we have*

- (1) $\text{Hom}_{\mathcal{E}}(C, Y_n) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{E})}(D^n(C), Y)$
- (2) $\text{Hom}_{\mathcal{E}}(X_{n-1}, C) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{E})}(X, D^n(C))$
- (3) If $\text{Ker}(d_n^Y)$ exists then $\text{Hom}_{\mathcal{E}}(C, \text{Ker}(d_n^Y)) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{E})}(S^n(C), Y)$
- (4) If $\text{Coker}(d_{n+1}^X)$ exists then $\text{Hom}_{\mathcal{E}}(\text{Coker}(d_{n+1}^X), C) \cong \text{Hom}_{\mathbf{Ch}(\mathcal{E})}(X, S^n(C))$
- (5) $\text{Ext}_{\mathcal{E}}^1(C, Y_n) \cong \text{Ext}_{\mathbf{Ch}(\mathcal{E})}^1(D^n C, Y)$
- (6) $\text{Ext}_{\mathcal{E}}^1(X_n, C) \cong \text{Ext}_{\mathbf{Ch}(\mathcal{E})}^1(X, D^{n+1} C)$

(7) Let X be a complex such that $\text{Ker}d_n^X$ exists. Then there is a monic

$$\text{Ext}^1(C, \text{Ker}(d_n^X)) \hookrightarrow \text{Ext}^1(S^n C, X)$$

If X is acyclic then this is an isomorphism.

(8) Let X be a complex such that $\text{Coker}(d_{n+1}^X)$ exists. Then there is a monic

$$\text{Ext}^1(\text{Coker}(d_{n+1}^X), C) \hookrightarrow \text{Ext}^1(X, S^n C)$$

If X is acyclic then this is an isomorphism.

Proof. By Proposition 2.43 and Proposition 2.62 it is sufficient to prove statements 1 – 3, 5, 6, 7 under the assumption that \mathcal{E} is abelian. In this context the result is Lemma 3.1 in [Gil04] and Lemma 4.2 in [Gil08]. Statement 4 is dual to 3, and statement 8 is dual to 7. \square

Remark 3.11. It is possible to prove most of this lemma internally in an exact category without passing to an abelianisation.

At this point we can prove the following lemma. It provides one of our main applications of generating sets, namely a convenient method for testing acyclicity. It is a modification of Lemma 3.7 in [Gil07].

Lemma 3.12. Let \mathcal{E} be an exact category with a set of generators \mathcal{G} . Let X be a chain complex. Suppose that X_\bullet is good. If for every $G \in \mathcal{G}$ each map $f : S^n(G) \rightarrow X$ extends to $D^{n+1}(G)$, then X is acyclic.

Proof. By Proposition 2.29 it is enough to show that whenever d_m has a kernel, the induced map

$$d' : X_{m+1} \rightarrow Z_m X$$

is an admissible epic. For this it is enough to show that for each $G \in \mathcal{G}$,

$$\text{Hom}(G, d') : \text{Hom}(G, X_{m+1}) \rightarrow \text{Hom}(G, Z_m X)$$

is surjective, i.e. that any map $f : G \rightarrow Z_m X$ lifts to a diagram

$$\begin{array}{ccc} & & X_{m+1} \\ & \nearrow & \downarrow d' \\ G & \xrightarrow{f} & Z_m X \end{array}$$

But this is equivalent to showing that the chain map $S^n(G) \rightarrow X$ induced by f extends to a morphism $D^{n+1}(G) \rightarrow X$. \square

Next we characterise projective objects in categories of chain complexes. It is well known that projective objects in the category of chain complexes in an abelian category are precisely the split exact complexes with projective entries. See for example [Hov07] Proposition 2.3.10. We generalise the result to exact categories.

Proposition 3.13. Let \mathcal{E} be a weakly idempotent complete exact category, and let $*$ $\in \{\geq 0, \leq 0, +, -, b, \emptyset\}$. Then split exact complexes of projectives are projective objects in $\mathbf{Ch}_*(\mathcal{E})$. In addition, if P is projective in \mathcal{E} then $S^0(P)$ is projective in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$.

Conversely, if a complex X_\bullet is a projective in $\mathbf{Ch}_*(\mathcal{E})$ for $*$ $\in \{+, -, b, \geq 0, \leq 0, \emptyset\}$ then every X_n is projective. Moreover, if $*$ $\in \{+, -, b, \emptyset\}$ and X_\bullet is good then X_\bullet is a split exact complex of projective objects of \mathcal{E} . In particular if \mathcal{E} has all kernels then the projective objects in $\mathbf{Ch}_*(\mathcal{E})$, for $*$ $\in \{+, -, b, \emptyset\}$ are precisely the split exact complexes of projectives contained in $\mathbf{Ch}_*(\mathcal{E})$.

Proof. By Lemma 3.10, split exact complexes of projectives are projective objects in $\mathbf{Ch}_*(\mathcal{E})$ for $*$ $\in \{+, -, b, \geq 0, \leq 0, \emptyset\}$. Let us show that $S^0(P)$ is a projective object in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ whenever P is projective in \mathcal{E} . Indeed in this case, Lemma 3.10 implies that $\mathrm{Hom}_{\mathbf{Ch}(\mathcal{E})}(S^0(P), Y_\bullet) \cong \mathrm{Hom}_{\mathcal{E}}(P, Y_0)$. Since P is projective, $S^0(P)$ is as well.

Conversely if a complex X_\bullet is a projective complex, then it follows immediately from Lemma 3.10 that each X_n is projective in \mathcal{E} . Suppose that $*$ $\in \{+, -, b, \emptyset\}$ and that X_\bullet is good. Let P be the cone of the identity $id : X_\bullet \rightarrow X_\bullet$. Consider the surjection $P \rightarrow X_\bullet[-1]$. Since $X_\bullet[-1]$ is projective this map splits by Proposition 2.48. The second factor of this splitting gives a homotopy between $id_{X_\bullet[-1]}$ and the 0 map. By Corollary 2.36, $X_\bullet[-1]$ is acyclic so X_\bullet is as well. \square

We can now show that $\mathbf{Ch}_*(\mathcal{E})$ has enough projectives. (This is well known for $\mathbf{Ch}(\mathcal{A})$ with \mathcal{A} abelian. See for example [Wei95] Exercise 2.2.2).

Corollary 3.14. *Let \mathcal{E} be an exact category with enough projectives. Then $\mathbf{Ch}_*(\mathcal{E})$ has enough projectives for $*$ $\in \{+, -, b, \leq 0, \geq 0, \emptyset\}$*

Proof. By Proposition 3.13 $D^n(P)$ is projective in $\mathbf{Ch}_*(\mathcal{E})$ for $*$ $\in \{+, -, b, \emptyset\}$ whenever P is projective. Also $D^n(P)$ for $n \leq 0$ is projective in $\mathbf{Ch}_{\leq 0}(\mathcal{E})$. Let $X_\bullet \in \mathbf{Ch}_*(\mathcal{E})$ for $*$ $\in \{+, -, b, \leq 0, \emptyset\}$. For each n pick a projective P_n and an admissible epimorphism $P_n \twoheadrightarrow X_n$. This induces a map $D^n(P_n) \rightarrow X_\bullet$ which is an admissible epimorphism in degree n . Let $P_\bullet = \bigoplus_n D^n(P_n)$. By the above discussion we have an admissible epimorphism $P_\bullet \twoheadrightarrow X_\bullet$.

Now let $X_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{E})$. For $n > 0$ the object $D^n(P)$ is projective in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$. $S^0(P)$ is also projective in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$. For $n > 0$, as before there is a projective object P_n and a morphism $D^n(P_n) \rightarrow X_\bullet$ which is an admissible epimorphism in degree n . For $n = 0$ pick a projective object P_0 and an admissible epimorphism $P_0 \rightarrow X_0$. Since $X_{-1} = 0$, this induces a map $S^0(P_0) \rightarrow X_\bullet$ which is an admissible epimorphism in degree 0. Let $P_\bullet = \left(\bigoplus_{n>0} D^n(P_n) \right) \oplus S^0(P_0)$. Then we have an admissible epimorphism $P_\bullet \twoheadrightarrow X_\bullet$. \square

We have essentially shown that $\mathbf{Ch}_*(\mathcal{E})$ has a set of projective generators whenever \mathcal{E} does.

Corollary 3.15. *Suppose \mathcal{P} is a set of admissible generators for an exact category \mathcal{E} . Then $D^*(\mathcal{P}) = \{D^n(P) : P \in \mathcal{P}, n \in \mathbb{Z}\} \cap \mathbf{Ch}_*(\mathcal{E})$ is a set of generators for $\mathbf{Ch}_*(\mathcal{E})$ and $*$ $\in \{+, -, b, \leq 0, \emptyset\}$. For $*$ $\in \{\geq 0\}$, $\tilde{D}^*(\mathcal{P}) := D^*(\mathcal{P}) \cup \{S^0(P) : P \in \mathcal{P}\}$ is a set of generators for $\mathbf{Ch}_*(\mathcal{E})$. They are projective generating sets if \mathcal{P} is.*

Proof. The proof of Corollary 3.14 shows that the sets in the statement of the proposition are admissible generating sets. Proposition 3.13 establishes the second assertion. \square

We are nearly ready to show that $\mathbf{Ch}_*(\mathcal{E})$ is elementary for $*$ $\in \{+, \geq 0, \leq 0, -, b, \emptyset\}$. It remains to identify some tiny objects in diagram categories.

Proposition 3.16. *Let \mathcal{E} be an additive category, \mathcal{J} a small directed set such that for each pair of objects j, j' in \mathcal{J} $\mathrm{Hom}(j, j')$ is finite (note it can be empty). Let \mathcal{M} be as in the statement of Corollary 2.59. Let $X \in [\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ be such that $X(j)$ is tiny for all j , and $X(j) = 0$ for all but finitely many j . Then X is a tiny object in $[\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$.*

Proof. Let $F : \mathcal{I} \rightarrow [\mathcal{J}, \mathcal{E}]_{\mathcal{M}}$ be a filtering inductive system. Denote the limit by L . Then $L(j) = \lim_{\rightarrow \mathcal{I}} F(j)$.

We have the commutative diagram

$$\begin{array}{ccccccc}
0 & \downarrow & & & 0 & \downarrow & \\
\text{Hom}(X, L) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \text{Hom}(X, F_i) & & \text{Hom}(X, L) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \text{Hom}(X, F_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_j \text{Hom}(X(j), L(j)) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \bigoplus_j \text{Hom}(X(j), F(j)_i) & & \bigoplus_j \text{Hom}(X(j), L(j)) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \bigoplus_j \text{Hom}(X(j), F(j)_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_j \left(\bigoplus_{m \leq j} \text{Hom}(X(m), L(j)) \oplus \bigoplus_{j \leq n} \text{Hom}(X(j), L(n)) \right) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \bigoplus_j \left(\bigoplus_{m \leq j} \text{Hom}(X(m), F(j)_i) \oplus \bigoplus_{j \leq n} \text{Hom}(X(j), F(n)_i) \right)
\end{array}$$

Now the first column is exact. Since direct limits of abelian groups are exact, the second column is also exact. The condition on \mathcal{I} implies that for each j , there are finite subsets $m(j)$ of $\{m : m \leq j\}$ and $n(j)$ of $\{n : n \geq j\}$ such that in the diagram

$$\begin{array}{ccccccc}
0 & \downarrow & & & 0 & \downarrow & \\
\text{Hom}(X, L) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \text{Hom}(X, F_i) & & \text{Hom}(X, L) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \text{Hom}(X, F_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_j \text{Hom}(X(j), L(j)) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \bigoplus_j \text{Hom}(X(j), F(j)_i) & & \bigoplus_j \text{Hom}(X(j), L(j)) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \bigoplus_j \text{Hom}(X(j), F(j)_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_j \left(\bigoplus_{m \in m(j)} \text{Hom}(X(m), L(j)) \oplus \bigoplus_{n \in n(j)} \text{Hom}(X(j), L(n)) \right) & \longrightarrow & \lim_{\rightarrow \mathcal{I}} \bigoplus_j \left(\bigoplus_{m \in m(j)} \text{Hom}(X(m), F(j)_i) \oplus \bigoplus_{n \in n(j)} \text{Hom}(X(j), F(n)_i) \right)
\end{array}$$

both columns are still exact.

Both the middle and bottom horizontal maps are isomorphisms because there are only finitely many non-zero X_j and they are tiny. Since both columns are exact, the top horizontal map must be an isomorphism as well. \square

As a consequence we have

Corollary 3.17. *Let \mathcal{E} be an elementary exact category. Then $\mathbf{Ch}_*(\mathcal{E})$ is elementary for $*$ $\in \{+, \leq, 0, \geq, 0, -, b, \emptyset\}$.*

Proof. Let \mathcal{P} be a projective generating set consisting of tiny objects. The sets $D^*(\mathcal{P})$ (resp. $\tilde{D}^*(\mathcal{P})$) are projective generating sets in $\mathbf{Ch}_*(\mathcal{E})$ for $*$ $\in \{\leq 0, +, -, b, \emptyset\}$ (resp. $*$ $\in \{\geq 0\}$). For each $n \in \mathbb{Z}$ $D^n(P)$ is tiny, as is $S^n(P)$, by Proposition 3.16. \square

3.3. Generators in Monoidal Exact Categories. Let us briefly mention a useful compatibility condition between generators and monoidal structure.

Definition 3.18. *A monoidal exact category which has a set of flat admissible generators is said to be **flatly generated**.*

Definition 3.19. *A projectively monoidal exact category which is also (quasi)-elementary is said to be **monoidal (quasi)-elementary***

Proposition 3.20. *Suppose that $(\mathcal{E}, \otimes, k)$ is a flatly generated monoidal exact category in which direct sums are exact. Then every projective object is flat.*

Proof. In this case every projective will be a summand of a flat object, and therefore flat. \square

In particular to check that a category is projectively monoidal, it suffices to find a set of flat generators.

3.4. Generators and Adjunctions. We conclude this section with a note about passing generating sets through adjunctions. The specific application we have in mind is to categories of algebras over compatible monads. We have the following general setup $F : \mathcal{E} \rightarrow \mathcal{D}$ and $|-| : \mathcal{D} \rightarrow \mathcal{E}$ are additive functors between exact categories. Moreover these functors form an adjoint pair

$$F \dashv |-|$$

Proposition 3.21. *Let $F \dashv |-|$ be an adjunction as above. Suppose that $|-|$ is an exact functor. If P is a projective object of \mathcal{E} then $F(P)$ is a projective object of \mathcal{D} .*

Proof. Let

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

be a short exact sequence in \mathcal{D} , and let P be projective in \mathcal{E} . Then we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(F(P), X) & \longrightarrow & \mathrm{Hom}(F(P), Y) & \longrightarrow & \mathrm{Hom}(F(P), Z) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}(P, |X|) & \longrightarrow & \mathrm{Hom}(P, |Y|) & \longrightarrow & \mathrm{Hom}(P, |Z|) \longrightarrow 0 \end{array}$$

The vertical arrows are isomorphisms and the bottom row is exact since $|-|$ is exact and P is projective. Hence the top row is short exact well. \square

We know how adjunctions act on projectives. Let us now see what happens on generating sets.

Proposition 3.22. *Let $F \dashv |-|$ be an adjunction as above. Suppose that $|-|$ reflects admissible epimorphisms, and that \mathcal{E} has an admissible generating set \mathcal{G} . Let $F(\mathcal{G})$ denote the collection $\{F(G) : G \in \mathcal{G}\}$ of objects of \mathcal{D} . Then $F(\mathcal{G})$ is an admissible generating set in \mathcal{D} .*

Proof. Let X be an object of \mathcal{D} . Suppose there is some object Q of \mathcal{E} and an admissible epimorphism $p : Q \rightarrow |X|$. There is an induced morphism $\tilde{p} : F(Q) \rightarrow X$. Then p coincides with the composition $Q \rightarrow |F(Q)| \rightarrow |X|$. By Proposition 2.9, the map $|\tilde{p}|$ is an admissible epimorphism. Since $|-|$ reflects admissible epimorphisms, \tilde{p} is an admissible epimorphism in \mathcal{D} .

Now let \mathcal{G} be an admissible generating set in \mathcal{E} , and let X be an object of \mathcal{D} . Since \mathcal{G} is an admissible generating set, there is an object G of $\bigoplus \mathcal{G}$ and an admissible epimorphism $G \rightarrow |X|$. The induced morphism $F(G) \rightarrow X$ is an admissible epimorphism by the above remarks. Since F is a left adjoint it preserves colimits, so $F(G)$ is an element of $\bigoplus F(\mathcal{G})$. \square

Proposition 3.23. *Let $F \dashv |-|$ be an adjunction as above.*

- (1) *Suppose that $|-|$ is exact and reflects admissible epimorphisms. If \mathcal{G} is a projective generating set in \mathcal{E} then $F(\mathcal{G})$ is a projective generating set in \mathcal{D} .*
- (2) *Suppose that $|-|$ is exact, reflects epimorphisms and preserves direct sums (resp. filtered colimits). If \mathcal{E} is quasi-elementary (resp. elementary) then so is \mathcal{D} .*

Proof. (1) The first assertion follows from Proposition 3.21 and Proposition 3.22.

- (2) Since $|-|$ preserves direct sums (resp. filtered colimits) it is clear that F preserves small (resp. tiny) objects. \square

Example 3.24. *Let T be a compatible monad on an exact category \mathcal{E} . Then the forgetful functor $|-| : \mathcal{E}^T \rightarrow \mathcal{E}$ has a right adjoint $F : \mathcal{E} \rightarrow \mathcal{E}^T$ assigning to an object the free T -algebra on it. By construction of the exact structure on \mathcal{E}^T in Proposition 2.73, the functor $|-|$ is admissibly exact and reflects exactness. Moreover it creates limits and colimits. By Lemma 2.74, Proposition 3.23 is applicable in such categories.*

3.5. Examples: $\mathbf{Ind}(\mathbf{Ban}_{\mathbb{C}})$ and $\mathbf{CBorn}_{\mathbb{C}}$. Let $\mathbf{Ban}_{\mathbb{C}}$ denote the category of Banach spaces over \mathbb{C} , i.e. the category whose objects are complete normed \mathbb{C} -vector spaces and whose morphisms are bounded maps. We also implicitly assume that we have fixed a universe \mathbb{U} , and that the underlying set of a Banach space is contained in \mathbb{U} . By [Pro00], this is a quasi-abelian category. Moreover it has enough projectives. In fact the projectives are precisely the spaces $l^1(I)$ of summable sequences indexed by some set I ([Pro95]).

$\mathbf{Ban}_{\mathbb{C}}$ also has a monoidal structure. If E and F are Banach spaces, the **projective tensor product** of E and F , denoted $E \hat{\otimes}_{\pi} F$ is the completion of the usual tensor product $E \otimes F$ with respect to the cross-norm

$$\|u\| = \inf \left\{ \sum_{i=1}^n \|e_i\| \|f_i\| : u = \sum_{i=1}^n e_i \otimes f_i \right\}$$

It is a closed monoidal structure. The vector space $\mathrm{Hom}_{\mathbb{C}}(E, F)$ of bounded maps between E and F can be given the structure of a Banach space. The norm of $T : E \rightarrow F$ is

$$\|T\| = \sup_{e \in E \setminus \{0\}} \frac{\|T(e)\|_F}{\|e\|_E}$$

This gives an internal Hom functor, which we denote by $\underline{\mathrm{Hom}}$. For details see [BBK13]. Thus $(\mathbf{Ban}_{\mathbb{C}}, \hat{\otimes}_{\pi}, \underline{\mathrm{Hom}})$ is a monoidal quasi-abelian category. Finally, the projective objects $l^1(I)$ are flat by [BBB15]. By Proposition 3.20 this category is projectively monoidal. Recall that if \mathcal{C} is a \mathbb{U} -small category for some universe \mathbb{U} , and \mathbb{V} a universe, then the \mathbb{V} -ind-completion of \mathcal{C} is a category constructed as follows. Objects are diagrams $E : \mathcal{I} \rightarrow \mathcal{C}$ where \mathcal{I} is a \mathbb{V} -small filtrant category. If $E : \mathcal{I} \rightarrow \mathcal{C}$ and $F : \mathcal{J} \rightarrow \mathcal{C}$ are objects in $\mathbf{Ind}(\mathcal{C})$ then we write

$$\mathrm{Hom}_{\mathbf{Ind}(\mathcal{C})}(E, F) = \lim_{\leftarrow \mathcal{I}} \lim_{\rightarrow \mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(E_i, F_j)$$

Proposition 3.25. *Let \mathcal{E} be a quasi-abelian category with enough projectives. Then $\mathbf{Ind}(\mathcal{E})$ is a cocomplete elementary quasi-abelian category. Moreover, if \mathcal{E} is a closed monoidal exact category, then its ind-completion*

has a canonical exact closed monoidal structure extending the one on \mathcal{E} . Finally if \mathcal{E} projectively monoidal $\mathbf{Ind}(\mathcal{E})$.

Proof. See [Sch99] Proposition 2.1.16 and Proposition 2.1.19. \square

Corollary 3.26. *The category $\mathbf{Ind}(\mathbf{Ban})$ is a locally presentable closed, monoidal elementary quasi-abelian category.*

Two more examples of closed, monoidal elementary quasi-abelian category are $\mathbf{Born}_{\mathbb{C}}$ $\mathbf{CBorn}_{\mathbb{C}}$, the categories of bornological spaces of convex type, and complete bornological spaces of convex type respectively. See [BBB15].

4. MODEL STRUCTURES ON EXACT CATEGORIES

In [Hov02], Hovey introduced the notion of a **compatible model structure** on an abelian category. He showed that there is a 1-1 correspondence between such model structures and purely homological data now known as **Hovey triples**. Gillespie noticed that this correspondence generalises to weakly idempotent complete exact categories, and explains in [Gil11] how to adapt Hovey's proofs. In the next two subsections we will recall some of Hovey's/ Gillespie's results both for the reader's convenience and because we will need many of the individual propositions later anyway. We shall modify the exposition somewhat, by first extracting from Hovey's proof a bijection between cotorsion pairs and compatible weak factorisation systems (this has been noticed in [Št'12]). For basic facts about weak factorisation systems and model structures in general see Appendix B.

4.1. Cotorsion Pairs and Weak Factorization Systems. Let \mathcal{S} be a class of objects in an exact category \mathcal{E} . We shall denote by ${}^{\perp}\mathcal{S}$ the class of all objects X such that $\mathrm{Ext}^1(X, S) = 0$ for all $S \in \mathcal{S}$, and by \mathcal{S}^{\perp} the class of all objects X such that $\mathrm{Ext}^1(S, X) = 0$ for all $S \in \mathcal{S}$. The class \mathcal{S}^{\perp} is called the class of \mathcal{S} -**injectives**, and the class ${}^{\perp}\mathcal{S}$ is called the class of \mathcal{S} -**projectives**. The following technical result will be useful. The proof is a straightforward generalisation of Lemma 6.2 in [Hov02].

Lemma 4.1. *Let \mathcal{E} be an exact category. Let \mathcal{S} be a class of objects in \mathcal{E} , and let $\mathfrak{L} = {}^{\perp}\mathcal{S}$. Then \mathfrak{L} is closed under retracts and finite extensions. If \mathcal{E} is cocomplete it is closed under transfinite extensions.*

Proof. First we show that \mathfrak{L} is closed under retracts. Note that it is sufficient to show that for a given $Y \in \mathcal{E}$, the collection of objects X such that $\mathrm{Ext}^1(X, Y) = 0$ is closed under retracts. Let X be such that $\mathrm{Ext}^1(X, Y) = 0$ and let X' be a retract of X . Then X' is a summand of X , and so $\mathrm{Ext}^1(X', Y) = 0$.

Let us show that \mathfrak{L} is closed under transfinite extensions. Again it is sufficient to show that for any object $Y \in \mathcal{E}$ the collection of all X with $\mathrm{Ext}^1(X, Y) = 0$ is closed under transfinite extensions and retracts. More generally, suppose ϕ is an ordinal such that for any $\phi' \leq \phi$, \mathcal{E} has ϕ' -colimits. Suppose λ is an ordinal with $\lambda \leq \phi$ and $X : \lambda \rightarrow \mathcal{E}$ is a colimit-preserving functor such that $\mathrm{Ext}^1(X_0, Y) = 0$, $X_{\alpha} \rightarrow X_{\alpha+1}$ is an admissible monic for all $\alpha < \lambda$ and $\mathrm{Ext}^1(\mathrm{Coker}(X_{\alpha} \rightarrow X_{\alpha+1}), Y) = 0$ for all $\alpha < \lambda$. We shall prove by transfinite induction that $\mathrm{Ext}^1(X_{\beta}, Y) = 0$ for all $\beta \leq \lambda$, where $X_{\lambda} = \mathrm{colim}_{\alpha < \lambda} X_{\alpha}$. If $\lambda = 0$ then this is clear. Suppose that λ is a successor ordinal, so that $\lambda = \alpha + 1$ for some ordinal α . We have $\mathrm{Ext}^1(X_{\alpha}, Y) = 0$ and $\mathrm{Ext}^1(\mathrm{Coker}(X_{\alpha} \rightarrow X_{\lambda}), Y) = 0$. The long exact Ext sequence then shows that $\mathrm{Ext}^1(X_{\lambda}, Y) = 0$.

Now suppose that $\beta \leq \lambda$ is a limit ordinal, and that $\mathrm{Ext}^1(X_{\alpha}, Y) = 0$ for all $\alpha < \beta$. Let

$$0 \longrightarrow Y \xrightarrow{f} N \xrightarrow{p} X_{\beta} \longrightarrow 0$$

represent an element of $\text{Ext}^1(X_\beta, Y)$. For each $\alpha \leq \beta$, pull this short exact sequence back through the map $j_\alpha : X_\alpha \rightarrow X_\beta$ for $\alpha \leq \beta$. For $\alpha \leq \gamma < \beta$ we get a commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{f} & N & \xrightarrow{p} & X_\beta & \longrightarrow & 0 \\
& & \parallel & & \uparrow k_\gamma & & \uparrow j_\gamma & & \\
0 & \longrightarrow & Y & \xrightarrow{f_\gamma} & N_\gamma & \xrightarrow{p_\gamma} & X_\gamma & \longrightarrow & 0 \\
& & \parallel & & \uparrow k_{\alpha,\gamma} & & \uparrow j_{\alpha,\gamma} & & \\
0 & \longrightarrow & Y & \xrightarrow{f_\alpha} & N_\alpha & \xrightarrow{p_\alpha} & X_\alpha & \longrightarrow & 0
\end{array}$$

Since f is an admissible monic, f_α is as well by Proposition 2.9. Since $\text{Ext}^1(X_\alpha, Y) = 0$ there is some splitting $t_\alpha : X_\alpha \rightarrow N_\alpha$ of p_α . We are going to modify the t_α to s_α so that they are compatible, i.e. $k_{\alpha,\gamma}s_\alpha = s_\gamma j_{\alpha,\gamma}$ for all $\alpha \leq \gamma$. We will do this by transfinite induction.

Set $s_0 = t_0$. If γ is a limit ordinal let $s_\gamma : \text{colim}_{\alpha < \gamma} X_\alpha \rightarrow N_\gamma$ be the map whose restriction to X_α is $k_{\alpha,\gamma}s_\alpha$, where $k_{\alpha,\gamma} : N_\alpha \rightarrow N_\gamma$ is the transfinite composition of the continuous functor $\gamma \rightarrow \mathcal{E}$, $\beta \mapsto N_\beta$. Then by construction $k_{\alpha,\gamma}s_\alpha = s_\gamma j_{\alpha,\gamma}$.

Now for the successor case. Suppose we have constructed s_α . Let us construct $s_{\alpha+1}$. We have

$$\begin{aligned}
p_{\alpha+1}(k_{\alpha,\alpha+1}s_\alpha - t_{\alpha+1}j_{\alpha,\alpha+1}) &= j_{\alpha,\alpha+1} \circ p_\alpha \circ s_\alpha - p_{\alpha+1} \circ t_{\alpha+1} \circ j_{\alpha,\alpha+1} \\
&= j_{\alpha,\alpha+1} - j_{\alpha,\alpha+1} \\
&= 0
\end{aligned}$$

Therefore there is a map $h : X_\alpha \rightarrow Y$ such that $f_{\alpha+1}h = k_{\alpha,\alpha+1}s_\alpha - t_{\alpha+1}j_{\alpha,\alpha+1}$. Since $j_{\alpha,\alpha+1} : X_\alpha \rightarrow X_{\alpha+1}$ is an admissible monic and $\text{Ext}^1(\text{Coker}(j_{\alpha,\alpha+1}), Y) = 0$, the long exact Ext sequence implies that there is a map $g : X_{\alpha+1} \rightarrow Y$ such that $gj_{\alpha,\alpha+1} = h$. Let $s_{\alpha+1} = t_{\alpha+1} + f_{\alpha+1}g$. Then clearly $s_{\alpha+1}$ is a section of $p_{\alpha+1}$. Moreover

$$\begin{aligned}
s_{\alpha+1}j_{\alpha,\alpha+1} &= t_{\alpha+1} \circ j_{\alpha,\alpha+1} + f_{\alpha+1}g \circ j_{\alpha,\alpha+1} \\
&= k_{\alpha,\alpha+1}s_\alpha - f_{\alpha+1} \circ h + f_{\alpha+1} \circ h \\
&= k_{\alpha,\alpha+1}s_\alpha
\end{aligned}$$

as required. \square

Let us now define cotorsion pairs, and discuss their relation with weak factorisation systems. We shall largely follow the notation of [Št'12].

Definition 4.2. Let \mathcal{E} be an exact category. A **cotorsion pair** on \mathcal{E} is a pair of families of objects $(\mathfrak{L}, \mathfrak{R})$ of \mathcal{E} such that $\mathfrak{L} = {}^\perp \mathfrak{R}$ and $\mathfrak{R} = \mathfrak{L}^\perp$.

Definition 4.3. A cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ is said to have **enough (functorial) projectives** if for every $X \in \mathcal{E}$ there is an admissible epic $p : Y \rightarrow X$, (functorial in X), such that $Y \in \mathfrak{L}$ and $\text{Ker}(p) \in \mathfrak{R}$. It is said to have **enough (functorial) injectives** if, for every X , there is an admissible monic $i : X \rightarrow Z$, (functorial in X), such that $Z \in \mathfrak{R}$ and $\text{Coker}(i) \in \mathfrak{L}$. A cotorsion pair is said to be **(functorially) complete** if it has enough (functorial) projectives and enough (functorial) injectives.

Example 4.4. Our main example is the projective cotorsion pair. Let \mathcal{E} be an exact category. Let $\mathbf{Proj}(\mathcal{E})$ denote the collection of projective objects of \mathcal{E} . Then $(\mathbf{Proj}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$ is clearly a cotorsion pair. Suppose that \mathcal{E} has enough (functorial) projectives. Then the cotorsion pair $(\mathbf{Proj}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$ is trivially (functorially) complete.

Notation 4.5. Let \mathcal{E} be an exact category and $(\mathcal{L}, \mathcal{R})$ a weak factorisation system on \mathcal{E} . Denote by $\text{Coker}\mathcal{L}$ the collection of objects L such that L is a cokernel of some map in \mathcal{L} and by $\text{Ker}\mathcal{R}$ the collection of objects R such that R is the kernel of some map in \mathcal{R} .

Given classes of objects $\mathfrak{A}, \mathfrak{B}$ in \mathcal{E} , we denote by $\text{Infl}(\mathfrak{A})$ the class of admissible monics with cokernel in \mathfrak{A} and by $\text{Defl}(\mathfrak{B})$ the class of admissible epics with kernel in \mathfrak{B} .

Definition 4.6. Let \mathcal{E} be an exact category. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ on \mathcal{E} is said to be **compatible** if

- (1) $f \in \mathcal{L}$ if and only if f is an admissible monic and $0 \rightarrow \text{Coker}(f)$ belongs to \mathcal{L} .
- (2) $f \in \mathcal{R}$ if and only if f is an admissible epic and $\text{Ker}(f) \rightarrow 0$ belongs to \mathcal{R} .

The following result is Theorem 5.13 in [Št'12].

Theorem 4.7. Let \mathcal{E} be an exact category. Then

$$(\mathcal{L}, \mathcal{R}) \mapsto (\text{Coker}\mathcal{L}, \text{Ker}\mathcal{R}) \text{ and } (\mathfrak{A}, \mathfrak{B}) \mapsto (\text{Infl}(\mathfrak{A}), \text{Defl}(\mathfrak{B}))$$

define mutually inverse bijective mappings between compatible weak factorisation systems and complete cotorsion pairs. The bijections restrict to mutually inverse mappings between compatible functorial weak factorisation systems and functorially complete cotorsion pairs.

4.2. Compatible Model Structures. Having described the bijection between cotorsion pairs and compatible weak factorisation systems, we now introduce compatible model structures, and explain how they too correspond to purely homological data. Remember that we do not assume our model categories are complete or cocomplete.

Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on an additive category \mathcal{E} .

Definition 4.8. Let \mathcal{E} be an exact category. Let $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ be a model structure on \mathcal{E} . The model structure is said to be **compatible** if both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are compatible weak factorisation systems.

Let us now define the corresponding homological data. As for abelian categories, we will call a subcategory \mathcal{D} of an exact category \mathcal{E} **thick** if whenever

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence and two of the objects are in \mathcal{D} , then so is the third.

Definition 4.9. A **Hovey triple** on an exact category \mathcal{E} is a triple $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ of collections of objects of \mathcal{E} such that the full subcategory on \mathfrak{W} is closed under retracts and thick, and that both $(\mathfrak{C}, \mathfrak{F} \cap \mathfrak{W})$ and $(\mathfrak{C} \cap \mathfrak{W}, \mathfrak{F})$ are complete cotorsion pairs.

We then have the following theorem (Theorem 6.9 in [Št'12]). It is originally due to [Hov02] in the abelian case and [Gil11] in the more general exact case.

Theorem 4.10. Let \mathcal{E} be a weakly idempotent complete exact category. Then there is a bijection between Hovey triples and compatible model structures. The correspondence assigns to a Hovey triple $(\mathfrak{C}, \mathfrak{W}, \mathfrak{F})$ the model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ such that

- (1) $\mathcal{C} = \text{Infl}(\mathfrak{C})$
- (2) $\mathcal{F} = \text{Defl}(\mathfrak{F})$
- (3) \mathcal{W} consists of morphisms of the form $p \circ i$ where $i \in \text{Infl}(\mathfrak{W})$ and $p \in \text{Defl}(\mathfrak{W})$.

Before we move on let us mention a more general notion than compatible model structures. We will need it when we consider the projective model structure on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$.

Definition 4.11. Let \mathcal{E} be an exact category. A model structure $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ on \mathcal{E} is said to be **left pseudo-compatible** if there are classes of objects \mathfrak{C} and \mathfrak{W} such that

(1) The full subcategory on \mathfrak{W} is thick.

(2) A map f is in \mathcal{C} (resp. $\mathcal{C} \cap \mathcal{W}$) if and only if it is an admissible monic with cokernel in \mathcal{C} (resp. $\mathcal{C} \cap \mathfrak{W}$).

(3) An admissible monic is in \mathcal{W} if and only if its cokernel is in \mathfrak{W} .

As before \mathcal{C}/\mathfrak{W} / $\mathcal{C} \cap \mathfrak{W}$ are called the **cofibrant** /*trivial*/ **trivially cofibrant** objects. The pair $(\mathcal{C}, \mathfrak{W})$ will be called the **left homological Waldhausen pair** of the model structure. Dually one defines **right pseudo-compatible** model structures and **right homological Waldhausen pairs**

Remark 4.12. The terminology comes from the notion of a Waldhausen category, in which classes of weak equivalences and cofibrations are specified.

Clearly any compatible model structure is left pseudo-compatible.

4.3. Small Cotorision Pairs and Cofibrant Generation. When working with model categories, it is computationally convenient that they be cofibrantly small/ cellular (see Appendix B for exactly what we mean here). In this section, we study what conditions on the cotorision pairs defining a compatible model structure guarantee that the model structure is cofibrantly small. The material here is adapted from [Hov02] §6 to exact categories.

Definition 4.13. Let \mathcal{E} be an exact category. A cotorision pair $(\mathfrak{L}, \mathfrak{R})$ on \mathcal{E} is said to be **cogenerated by a set** if there is a set of objects \mathcal{G} in \mathfrak{L} such that $X \in \mathfrak{R}$ if and only if $\text{Ext}^1(G, X) = 0$ for all $G \in \mathcal{G}$.

Definition 4.14. Suppose \mathcal{E} is an exact category. A cotorision pair $(\mathfrak{L}, \mathfrak{R})$ is said to be **small** if the following conditions hold

(1) \mathfrak{L} contains a set of admissible generators.

(2) $(\mathfrak{L}, \mathfrak{R})$ is cogenerated by a set \mathcal{G} .

(3) For each $G \in \mathcal{G}$ there is an admissible monic i_G with cokernel G such that, if $\text{Hom}_{\mathcal{E}}(i_G, X)$ is surjective for all $G \in \mathcal{G}$, then $X \in \mathfrak{R}$.

The set of i_G together with the maps $0 \rightarrow U_i$ for some generating set $\{U_i\}$ contained in \mathfrak{L} is called a set of **generating morphisms** of $(\mathfrak{L}, \mathfrak{R})$.

There is an easy example.

Example 4.15. Recall the projective cotorision pair $(\mathbf{Proj}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$. Suppose that the category \mathcal{E} is projectively generated, with \mathcal{P} a generating set of projectives. We claim that in this case the projective cotorision pair is small. Indeed by assumption $\mathbf{Proj}(\mathcal{E})$ contains a set of generators \mathcal{P} . This set trivially cogenerated the cotorision pair as well. The third condition is also trivial.

We now come to the connection between cofibrantly small model structures and cotorision pairs. The proof of the following is a straightforward modification of [Hov02] Lemma 6.7.

Lemma 4.16. Let \mathcal{E} be an exact category together with a compatible weak factorisation system $(\mathcal{L}, \mathcal{R})$ with corresponding cotorision pair $(\mathfrak{L}, \mathfrak{R})$. If the cotorision pair is small, then this weak factorisation system is cofibrantly small. If in addition the generating morphisms have tiny domain, the weak factorisation system is cellular.

Proof. Let I denote a set of generating morphisms for $(\mathfrak{L}, \mathfrak{R})$. Note that every map of I is an admissible monic with cokernel in \mathfrak{L} , i.e. it is in \mathcal{L} . Suppose $p : A \rightarrow B$ has the right lifting property with respect to I . Then $\text{Hom}(U_i, p)$ is surjective for every U_i in some generating set $\{U_i\}$ of \mathcal{E} , so p is an admissible epic. Now, we have a pull-back square

$$\begin{array}{ccc} \text{Ker}(p) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ A & \xrightarrow{p} & B \end{array}$$

By Lemma B.2 $\text{Ker}(p) \rightarrow 0$ also has the right lifting property with respect to I . Hence $\text{Hom}(i, \text{Ker}(p))$ is surjective for all $i \in I$. Therefore $\text{Ker}(p) \in \mathfrak{R}$, so p is in \mathcal{R} . The statement about cellularity is now obvious. \square

4.4. Cotorsion Pairs on Monoidal Exact Categories. In this section $(\mathcal{E}, \otimes, k)$ is a monoidal exact category.

We will now study sufficient conditions on cotorsion pairs defining a model category structure so that the resulting structure is monoidal. We generalise the work of [Hov02] §7 to exact categories.

Definition 4.17. A short exact sequence in a monoidal exact category \mathcal{E} is said to be **pure** if it remains exact after tensoring with any object of \mathcal{E} . An admissible monic is said to be **pure** if it remains an admissible monic after tensoring with any object of \mathcal{E} .

Theorem 4.18. Let \mathcal{E} be a closed symmetric monoidal exact category. Suppose that \mathcal{E} has a left pseudo-compatible model structure with Waldhausen pair $(\mathfrak{C}, \mathfrak{W})$. Suppose the following conditions are satisfied.

- (1) Every cofibration is pure.
- (2) If $X, Y \in \mathfrak{C}$ then $X \otimes Y \in \mathfrak{C}$.
- (3) If $X, Y \in \mathfrak{C}$ and one of them is in \mathfrak{W} , then $X \otimes Y \in \mathfrak{C} \cap \mathfrak{W}$.
- (4) The unit I of the monoidal structure is in \mathfrak{C} .

Then \mathcal{E} is a monoidal model category.

In order to prove this we need the following result and its corollary.

Proposition 4.19. Let \mathcal{E} be a weakly idempotent complete exact category, Suppose we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{h} & Y & \xrightarrow{i} & Z & \longrightarrow & 0 \\ & & \downarrow \delta & & \downarrow \epsilon & & \downarrow \phi & & \\ 0 & \longrightarrow & P & \xrightarrow{j} & Q & \xrightarrow{k} & R & \longrightarrow & 0 \end{array}$$

with the top and bottom rows being short exact and the vertical arrows being admissible morphisms. Then there is an exact sequence

$$0 \rightarrow \text{Ker}(\delta) \rightarrow \text{Ker}(\epsilon) \rightarrow \text{Ker}(\phi) \rightarrow \text{Coker}(\delta) \rightarrow \text{Coker}(\epsilon) \rightarrow \text{Coker}(\phi) \rightarrow 0$$

Proof. Pass to a left abelianisation and use the Snake Lemma there. (This can actually be proved internally without passing to an abelianization, see [Büh10]). \square

Corollary 4.20. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{h} & Y & \xrightarrow{i} & Z & \longrightarrow & 0 \\ & & \downarrow \delta & & \downarrow \epsilon & & \downarrow \phi & & \\ 0 & \longrightarrow & P & \xrightarrow{j} & Q & \xrightarrow{k} & R & \longrightarrow & 0 \end{array}$$

be a commutative diagram with short-exact rows. Suppose that the map $\phi : Z \rightarrow R$ is an admissible monomorphism with cokernel $l : R \rightarrow S$ and that δ is an isomorphism. Then $\epsilon : Y \rightarrow Q$ is an admissible monomorphism with cokernel $l \circ k : Q \rightarrow S$.

Proof of Theorem 4.18. Let $i : A \rightarrow B$ and $j : A' \rightarrow B'$ be cofibrations with respective cokernels $f : B \rightarrow C$ and $g : B' \rightarrow C$. Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & A \otimes A' & \xrightarrow{1 \otimes j} & A \otimes B' & \xrightarrow{1 \otimes g} & A \otimes C' \longrightarrow 0 \\
& & \downarrow i \otimes 1 & & \downarrow & & \parallel \\
0 & \longrightarrow & B \otimes A' & \longrightarrow & P & \longrightarrow & A \otimes C' \longrightarrow 0 \\
& & \parallel & & \downarrow i \boxtimes j & & \downarrow i \otimes 1 \\
0 & \longrightarrow & B \otimes A' & \xrightarrow{1 \otimes j} & B \otimes B' & \xrightarrow{1 \otimes g} & B \otimes C' \longrightarrow 0
\end{array}$$

where the top left square is a push-out. Since cofibrations are pure by assumption, the rows of the diagram are exact. Moreover, both $i \otimes id_{A'}$ and $i \otimes id_{C'}$ are admissible monomorphisms, and the cokernel of $i \otimes id_{C'}$ is $C \otimes C'$. By Corollary 4.20, $i \boxtimes j$ is an admissible monomorphism with cokernel $C \otimes C'$. By assumption $C \otimes C' \in \mathfrak{C}$, so that $i \boxtimes j$ is a cofibration. Again by assumption, if either of C or C' is in \mathfrak{W} then so is $C \otimes C'$, and hence in this case $i \boxtimes j$ is a trivial cofibration. \square

Remark 4.21. *The statement of Theorem 4.18 also holds without the assumption that the monoidal structure is compatible with the exact structure, since it was not used at all in proof. This is also shown in [St'12]. However the remaining results do require this assumption.*

The next lemma says that if cofibrant objects are flat then condition 1 in Theorem 4.18 is automatically satisfied.

Lemma 4.22. *Suppose \mathcal{E} is a symmetric monoidal exact category with enough flat objects. If $C \in \mathcal{E}$ is flat then every short exact sequence*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is pure.

Proof. Suppose Z is arbitrary and let

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

be a short exact sequence with Y flat. We have a diagram

$$\begin{array}{ccccccc}
A \otimes X & \longrightarrow & A \otimes Y & \longrightarrow & A \otimes Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
B \otimes X & \longrightarrow & B \otimes Y & \longrightarrow & B \otimes Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
C \otimes X & \longrightarrow & C \otimes Y & \longrightarrow & C \otimes Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

with admissibly coacyclic rows and columns. The bottom row is short exact since C is flat. Since Y is flat the middle column is short exact. We need to prove that the right-hand column is short exact. In order to do this we may pass to a right abelianization of \mathcal{E} , and so without loss of generality assume that \mathcal{E} is abelian. Then the argument becomes a simple diagram chase. \square

Proposition 4.23. *Pure monics are stable under push out.*

Proof. Let $i : A \rightarrow B$ be a pure monic. Consider a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Since tensoring with Z preserves colimits,

$$\begin{array}{ccc} A \otimes Z & \longrightarrow & B \otimes Z \\ \downarrow & & \downarrow \\ X \otimes Z & \longrightarrow & Y \otimes Z \end{array}$$

is a push out. But by assumption $A \otimes Z \rightarrow B \otimes Z$ is an admissible monic. Hence $X \otimes Z \rightarrow Y \otimes Z$ is also an admissible monic. \square

Theorem 4.24. *Let \mathcal{E} be a bicomplete, monoidal exact category. Suppose that \mathcal{E} has a left pseudo-compatible model structure satisfying the hypotheses of Theorem 4.18. In addition, suppose that the following conditions hold*

(1) *If $X \in \mathfrak{C} \cap \mathfrak{W}$ and Y is arbitrary, then $X \otimes Y$ is in \mathfrak{W} .*

(2) *Transfinite compositions of weak equivalences which are also pure monics are still weak equivalences.*

Then the model structure satisfies the monoid axiom.

Proof. The first condition implies that if i is an acyclic cofibration, then $i \otimes Y$ is a weak equivalence. By Propositions 4.23 and Proposition 2.7, any push out of $i \otimes Y$ is a weak equivalence as well as a pure monic. By the second condition, any transfinite composition of such maps is a weak equivalence. \square

If in \mathcal{E} transfinite compositions of admissible monics are admissible monics (e.g. if \mathcal{E} is elementary) then one can replace the second condition by requiring that the class \mathfrak{W} is closed under transfinite compositions of pure monomorphisms. By this we mean that if λ is some ordinal, and $X : \lambda \rightarrow \mathcal{E}$ a continuous functor such that $0 \rightarrow X_0$ is a weak equivalence, and for each $i < j$ in λ the map $X_i \rightarrow X_j$ is a pure monic which is also a weak equivalence, then X_λ is in \mathfrak{W} . (This is the condition used in [Hov02] Theorem 7.4). Since \mathfrak{W} forms a thick subcategory and $X_0 \rightarrow X_\lambda$ is an admissible monic, this is equivalent to the cokernel of the map $X_0 \rightarrow X_\lambda$ being in \mathfrak{W} which in turn is equivalent to $X_0 \rightarrow X_\lambda$ being a weak equivalence.

4.5. Model Structures on Chain Complexes. Generalising results of [Gil04], in this section we describe a method for constructing compatible model structures on categories of chain complexes $\mathbf{Ch}_*(\mathcal{E})$ from cotorsion pairs on \mathcal{E} . Note that what we describe below will not always produce a model structure. However we will show in the next chapter that it does in the case that \mathcal{E} has enough projectives, and the cotorsion pair is the projective one (Example 4.4). First we define the collections of objects which will be candidates for the (trivially) fibrant and (trivially) cofibrant objects.

Definition 4.25. *Let $(\mathfrak{L}, \mathfrak{R})$ be a cotorsion pair on an exact category \mathcal{E} . Let $X \in \mathbf{Ch}(\mathcal{E})$ be a chain complex.*

(1) *X is called an \mathfrak{L} complex if it is acyclic and $Z_n X \in \mathfrak{L}$ for all n . The collection of all \mathfrak{L} complexes is denoted $\tilde{\mathfrak{L}}$.*

(2) *X is called an \mathfrak{R} complex if it is acyclic and $Z_n X \in \mathfrak{R}$ for all n . The collection of all \mathfrak{R} complexes is denoted $\tilde{\mathfrak{R}}$.*

(3) *X is called a $dg\mathfrak{L}$ complex if $X_n \in \mathfrak{L}$ for each n , and $\mathbf{Hom}(X, B)$ is exact whenever B is an \mathfrak{R} complex. The collection of all $dg\mathfrak{L}$ complexes is denoted $dg\tilde{\mathfrak{L}}$.*

- (4) X is called a $dg\mathfrak{R}$ complex if $X_n \in \mathfrak{R}$ for each n , and $\mathbf{Hom}(A, X)$ is exact whenever A is an \mathfrak{L} complex. The collection of all $dg\mathfrak{R}$ complexes is denoted $dg\tilde{\mathfrak{R}}$.

Notation 4.26. We define the collections $\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}}, dg\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}}$ similarly in the categories $\mathbf{Ch}_*(\mathcal{E})$ for $*$ $\in \{\geq, 0, \leq, +, -, b, \emptyset\}$. We will use the same notation for these collections irrespective of which category of chain complexes we are working in.

Remark 4.27. In $\mathbf{Ch}_*(\mathcal{E})$ for $*$ $\in \{+, -, \geq 0, b, \emptyset\}$ all of the above classes are closed under shifts $[n]$ for $n \leq 0$. For $*$ $\in \{+, -, \leq 0, b, \emptyset\}$ they are closed under shifts $[n]$ for $n \geq 0$.

Let us start to populate these collections. We first make the following easy observation.

Proposition 4.28. Let X be an \mathfrak{R} -complex. Then $X_n \in \mathfrak{R}$ for each n .

Proof. For each n we have a short exact sequence

$$0 \rightarrow Z_n X \rightarrow X_n \rightarrow Z_{n-1} X \rightarrow 0$$

and $Z_n X, Z_{n-1} X \in \mathfrak{R}$. By Proposition 4.1 \mathfrak{R} is closed under extensions. \square

With this in mind we have the following straightforward generalisation of [Gil04] Lemma 3.4.

Lemma 4.29. (1) Bounded below complexes with entries in \mathfrak{L} are $dg\mathfrak{L}$ complexes.

(2) Bounded above complex with entries in \mathfrak{R} are $dg\mathfrak{R}$ complexes.

Proof. (1) Let (X, d^X) be a bounded below complex with entries in \mathfrak{L} . We shall show that any $f : X \rightarrow B$ is homotopic to zero, where B is an \mathfrak{R} -complex. Consider such a map. Without loss of generality we may assume $X_n = 0$ for $n < 0$. Since d_1 is admissible, we have an exact sequence

$$0 \rightarrow \text{Ker} d_1^B \rightarrow B_1 \rightarrow \text{Im} d_1^B \rightarrow 0$$

Moreover, each object belongs to \mathfrak{R} . Since $X_{-1} = 0$, $f_0 : X_0 \rightarrow B_0$ factors through $\text{Im} d_1^B$. By abuse of notation, we also denote the factor $X_0 \rightarrow \text{Im} d_1^B$ by f_0 . By assumption, we have an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{E}}(X_0, \text{Ker} d_1^B) \rightarrow \text{Hom}_{\mathcal{E}}(X_0, B_1) \rightarrow \text{Hom}_{\mathcal{E}}(X_0, \text{Im} d_1^B) \rightarrow 0$$

Hence f_0 lifts to a map $D_1 : X_0 \rightarrow B_1$, i.e. there exists a D_1 such that $d_1^B D_1 = f_0$. Set $g_1 = f_1 - D_1 d_1^B$. Then $d_1^B g_1 = 0$ by direct calculation, so $g_1 : X_1 \rightarrow B_1$ factors through $\text{Im} d_2^B$. As before we can lift g_1 to a map $D_2 : X_1 \rightarrow B_2$ such that $d_2^B D_2 = g_1 = f_1 - D_1 d_1^B$. Continuing inductively, we construct a homotopy $\{D_k\}$ such that $d_k^B D_k + D_{k-1} d_{k-1}^X = f_{k-1}$ as required.

(2) This is dual to the previous part. \square

Gillespie's crucial Proposition 3.6 in [Gil04] does not hold in arbitrary exact categories. However some of it can be salvaged to give the following result.

Proposition 4.30. Let $(\mathfrak{L}, \mathfrak{R})$ be a cotorsion pair in an exact category \mathcal{E} . Then in $\mathbf{Ch}_*(\mathcal{E})$ for $*$ $\in \{+, -, b, \emptyset\}$ we have

(1) $dg\tilde{\mathfrak{L}} = {}^\perp \tilde{\mathfrak{R}}$.

(2) $dg\tilde{\mathfrak{R}} = \tilde{\mathfrak{L}}^\perp$

(3) $\tilde{\mathfrak{R}} \subseteq (dg\tilde{\mathfrak{L}})^\perp$

(4) $\tilde{\mathfrak{L}} \subseteq {}^\perp (dg\tilde{\mathfrak{R}})$

(5) Suppose \mathcal{E} has enough \mathcal{L} -objects. Let $X \in (dg\tilde{\mathcal{L}})^\perp$ be good. Then X is an \mathfrak{R} -complex.

(6) Suppose \mathcal{E} has enough \mathfrak{R} -objects. Let $X \in {}^\perp dg(\tilde{\mathfrak{R}})$ be cogood. Then X is an \mathcal{L} -complex.

Proof. (1) Let $X \in {}^\perp \tilde{\mathfrak{R}}$. Then $\text{Ext}^1(X, B) = 0$ whenever B is an \mathfrak{R} complex. In particular $\text{Ext}_{dw}^1(X, B) = 0$. Hence $\mathbf{Hom}(X, B)$ is exact whenever B is an \mathfrak{R} complex by Corollary 2.65. It remains to show $X_n \in \mathcal{L}$. Let $B \in \mathfrak{R}$. By Lemma 3.10 we have

$$\text{Ext}^1(X_n, B) = \text{Ext}^1(X, D^{n+1}B) = 0$$

since $D^{n+1}B \in \tilde{\mathfrak{R}}$. So $X_n \in \mathcal{L}$, and ${}^\perp \tilde{\mathfrak{R}} \subset dg\tilde{\mathcal{L}}$. Now let $X \in dg\tilde{\mathcal{L}}$. Since the entries of X are in \mathcal{L} , for any $Y \in \tilde{\mathfrak{R}}$, any short exact sequence

$$0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$$

is split exact in each degree. But also $\text{Ext}_{dw}^1(X, Y) = 0$. Hence, any sequence as above must be split exact, i.e. $\text{Ext}^1(X, Y) = 0$.

(2) This is dual to the previous part.

(3) Let $X \in \tilde{\mathfrak{R}}$ and $A \in dg\tilde{\mathcal{L}}$. Note that since $X_n \in \mathfrak{R}$, $\text{Ext}^1(X, A) = \text{Ext}_{dw}^1(X, A)$. Now since $\mathbf{Hom}(A, X)$ is exact, $\text{Ext}_{dw}^1(X, A) = 0$.

(4) This is dual to the previous part.

(5) Let us show that X is acyclic. We will again use Proposition 2.29. Let n be such that d_n has a kernel. Since we have enough \mathcal{L} -objects, we may choose an admissible epic $f'_n : A' \rightarrow Z_n X$ for some $A' \in \mathcal{L}$. By Lemma 3.10 this induces a map $f : S^n(A') \rightarrow X$. Now $\text{Ext}_{dw}^1(S^n(A')[-1], X) \subset \text{Ext}^1(S^n(A')[-1], X) = 0$ by assumption. Hence f is homotopic to 0. Applying Proposition 2.33 the map $d'_{n+1} : X_{n+1} \rightarrow Z_n X$ is an admissible epic. By Proposition 2.29 X is acyclic. To see that $Z_n X \in \mathfrak{R}$, we note that since X is acyclic, we have for any $A \in \mathcal{L}$,

$$\text{Ext}_X^1(A, Z_n X) \cong \text{Ext}^1(S^n A, X) = 0$$

Since $(\mathcal{L}, \mathfrak{R})$ is a cotorsion pair, $Z_n X \in \mathfrak{R}$. Hence $X \in \tilde{\mathfrak{R}}$ and so $(dg\tilde{\mathcal{L}})^\perp \subseteq \tilde{\mathfrak{R}}$.

(6) The proof for the second part is dual. □

We also have the following

Proposition 4.31. *Let $* \in \{\geq 0\}$, and let $(\mathcal{L}, \mathfrak{R})$ be a cotorsion pair in \mathcal{E} with enough \mathcal{L} -objects. Then $dg\tilde{\mathcal{L}} = {}^\perp \tilde{\mathfrak{R}}$ and $\tilde{\mathfrak{R}} = (dg\tilde{\mathcal{L}})^\perp$.*

Dually, if the cotorsion pair has enough \mathfrak{R} -objects, then for $ \in \{\leq 0\}$ $dg\tilde{\mathfrak{R}} = \tilde{\mathcal{L}}^\perp$ and $\tilde{\mathcal{L}} = {}^\perp dg(\tilde{\mathfrak{R}})$.*

Proof. The proofs of parts (3) and (5) in the previous proposition go through here, as does the proof that $dg\tilde{\mathcal{L}} \subset {}^\perp \tilde{\mathfrak{R}}$. Now let $X \in {}^\perp \tilde{\mathfrak{R}}$. The same proof as in part (1) of the previous proposition shows that each X_n must be an object in \mathcal{L} . Thus X is a bounded below complex of objects in \mathcal{L} and hence a $dg\tilde{\mathcal{L}}$ complex. □

We get as an immediate corollary:

Corollary 4.32. *Let $(\mathcal{L}, \mathfrak{R})$ be a cotorsion pair on an exact category \mathcal{E} with enough \mathcal{L} -objects and enough \mathfrak{R} -objects.*

(1) *$(dg\tilde{\mathcal{L}}, \tilde{\mathfrak{R}})$ is a cotorsion pair on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ and $\mathbf{Ch}_+(\mathcal{E})$. If \mathcal{E} has all kernels then it is a cotorsion pair on $\mathbf{Ch}(\mathcal{E})$.*

- (2) $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ is a cotorsion pair on $\mathbf{Ch}_{\leq 0}(\mathcal{E})$ and $\mathbf{Ch}_-(\mathcal{E})$. If \mathcal{E} has all cokernels then it is a cotorsion pair in $\mathbf{Ch}(\mathcal{E})$.
- (3) $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ and $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ are cotorsion pairs in $\mathbf{Ch}_b(\mathcal{E})$.
- (4) If \mathcal{E} has all kernels and cokernels, in particular if \mathcal{E} is quasi-abelian, then $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ and $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ are cotorsion pairs in $\mathbf{Ch}(\mathcal{E})$.

4.5.1. *Existence of dg-Model Structures.* The hope now is that the class \mathfrak{W} of acyclic complexes satisfies

$$\tilde{\mathfrak{L}} = dg\tilde{\mathfrak{L}} \cap \mathfrak{W}, \quad \tilde{\mathfrak{R}} = dg\tilde{\mathfrak{R}} \cap \mathfrak{W}$$

and that the cotorsion pairs $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ and $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ are functorially complete. It is not at all clear that this will be the case. In [YD14] it is shown that for a bicomplete abelian category in which infinite products are exact (i.e. an $AB4^*$ abelian category) it is always the case. We suspect this result can be easily adapted for bicomplete exact categories satisfying a similar condition. In general we do not know how to give useable conditions on a cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ which guarantee that $(dg\mathfrak{L}, \mathfrak{R})$ and $(\mathfrak{L}, dg\mathfrak{R})$ induce a model structure. However we will obtain some partial results in this direction. First we need acyclic complexes to form a thick subcategory.

Proposition 4.33. *Let \mathcal{E} be an exact category. Then for $*$ $\in \{\geq 0, \leq 0, +, -, b\}$ the full subcategory on \mathfrak{W} is a thick subcategory of $\mathbf{Ch}_*(\mathcal{E})$. If \mathcal{E} has all kernels then this is also true for $*$ $= \{\emptyset\}$.*

Proof. One may assume that \mathcal{E} is abelian by passing to a left abelianization for $*$ $\in \{\geq 0, +, b\}$, (or a right abelianization for $*$ $\in \{\leq 0, -\}$). The result in this case follows from the long exact sequence on homology. \square

It turns out that we always have the inclusions $\tilde{\mathfrak{L}} \subset dg\tilde{\mathfrak{L}} \cap \mathfrak{W}$, and $\tilde{\mathfrak{R}} \subset dg\tilde{\mathfrak{R}} \cap \mathfrak{W}$. This follows from the next result, which is an easy modification of the proof of [Gil04] Lemma 3.9.

Lemma 4.34. *Every chain map from an \mathfrak{L} complex to an \mathfrak{R} complex is homotopic to 0.*

Proof. Let X be an \mathfrak{L} -complex and Y an \mathfrak{R} -complex. Let $f : X \rightarrow Y$ be a chain map. Let us first show that f may be replaced with a homotopic map g which satisfies $d_n g_n = 0$, and $g_n d_{n+1} = 0$. We then show that such a map must necessarily be homotopic to 0.

The map $f_n : X_n \rightarrow Y_n$ restricts to $\hat{f}_n : Z_n X \rightarrow Z_n Y$. Moreover, since Y is acyclic we have an exact sequence

$$0 \rightarrow Z_{n+1} Y \rightarrow Y_{n+1} \rightarrow Z_n Y \rightarrow 0$$

and the objects in the sequence are in \mathfrak{R} . Since $Z_n X \in \mathfrak{L}$, we get an exact sequence

$$0 \rightarrow \text{Hom}(Z_n X, Z_{n+1} Y) \rightarrow \text{Hom}(Z_n X, Y_{n+1}) \rightarrow \text{Hom}(Z_n X, Z_n Y) \rightarrow 0$$

Hence there exists an $\alpha_n : Z_n X \rightarrow Y_{n+1}$ such that $d_{n+1} \alpha_n = \hat{f}_n$. We also know that

$$0 \rightarrow Z_n X \rightarrow X_n \rightarrow Z_{n-1} X \rightarrow 0$$

is an exact sequence of objects in \mathfrak{L} . So

$$0 \rightarrow \text{Hom}(Z_{n-1} X, Y_{n+1}) \rightarrow \text{Hom}(X_n, Y_{n+1}) \rightarrow \text{Hom}(Z_n X, Y_{n+1}) \rightarrow 0$$

is exact. Hence there exists $\beta_n : X_n \rightarrow Y_{n+1}$ which is α_n when restricted to $Z_n X$. Now put $g_n = f_n - (d_{n+1} \beta_n + \beta_{n-1} d_n)$. By direct calculation, $g = \{g_n\}$ is a chain map such that $d_n g_n = 0$ and $g_n d_{n+1} = 0$. By construction it is homotopic to f .

Now we show that a map g satisfying $d_n g_n = 0, g_n d_{n+1} = 0$ must be homotopic to 0. For convenience

we write $X_n/Z_nX = \text{Coker}(Z_nX \rightarrow X_n)$. Now g_n induces a map $\bar{g}_n : X_n/Z_nX \rightarrow Z_nY$ making the following diagram commute

$$\begin{array}{ccccc} X_n & \xleftarrow{=} & X_n & \xrightarrow{=} & X_n \\ \downarrow & & \downarrow \pi & & \downarrow g_n \\ Z_{n-1}X & \xleftarrow[\bar{d}_n]{\cong} & X_n/Z_nX & \xrightarrow{\bar{g}_n} & Z_nY \end{array}$$

Set $\hat{g}_n = \bar{g}_n \bar{d}_n^{-1} : Z_{n-1}X \rightarrow Z_nY$, so that $\hat{g}_n d_n = g_n$. Now, we have an exact sequence

$$0 \rightarrow \text{Hom}(Z_{n-1}X, Z_{n+1}Y) \rightarrow \text{Hom}(Z_{n-1}X, Y_{n+1}) \rightarrow \text{Hom}(Z_{n-1}X, Z_nY) \rightarrow 0$$

Thus there exists a map $\delta_n : Z_{n-1}X \rightarrow Y_{n+1}$ such that $d_{n+1}\delta_n = \hat{g}_n$. We claim that the maps $\delta_n d_n : X_n \rightarrow Y_{n+1}$ are a homotopy from g to 0. Indeed we have

$$\begin{aligned} d_{n+1}^Y \circ \delta_n \circ d_n^X - \delta_{n+1} \circ d_{n+1}^X \circ d_n^X &= g_n - 0 \\ &= g_n \end{aligned}$$

□

Corollary 4.35. *Let $(\mathfrak{L}, \mathfrak{R})$ be a cotorsion pair in an exact category. Then $\tilde{\mathfrak{L}} \subset dg\tilde{\mathfrak{L}} \cap \mathfrak{W}$, and $\tilde{\mathfrak{R}} \subset dg\tilde{\mathfrak{R}} \cap \mathfrak{W}$.*

In order to have any chance of getting the reverse inclusion, we'll need the cotorsion pair on \mathcal{E} to be hereditary. The following definition and the subsequent proposition are immediate generalisations of [Roz99] §1.2.3 from abelian categories to exact categories.

Definition 4.36. *A cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ is said to be **hereditary** if*

$$\text{Ext}^i(A, B) = 0$$

for any $A \in \mathfrak{L}, B \in \mathfrak{R}$ and $i \geq 1$.

Example 4.37. *Clearly the projective cotorsion pair is hereditary.*

Proposition 4.38. *Let $(\mathfrak{L}, \mathfrak{R})$ be a hereditary cotorsion pair on an exact category \mathcal{E} . Then*

(1) \mathfrak{L} is resolving. That is \mathfrak{L} is closed under taking kernels of admissible epis.

(2) \mathfrak{R} is coresolving. That is \mathfrak{R} is closed under taking cokernels of admissible monics.

If \mathcal{E} has enough \mathfrak{R} -projectives then $(\mathfrak{L}, \mathfrak{R})$ is hereditary if and only if \mathfrak{L} is resolving. Dually if \mathcal{E} has enough \mathfrak{L} -injectives then $(\mathfrak{L}, \mathfrak{R})$ is hereditary if and only if \mathfrak{R} is coresolving.

Proof. Let

$$0 \rightarrow K \rightarrow A \rightarrow A' \rightarrow 0$$

be an exact sequence with $A, A' \in \mathfrak{L}$. Let $B \in \mathfrak{R}$. Then we have the long exact sequence

$$\dots \rightarrow \text{Ext}^i(A', B) \rightarrow \text{Ext}^i(A, B) \rightarrow \text{Ext}^i(K, B) \rightarrow \text{Ext}^{i+1}(A', B) \rightarrow \dots$$

Now for $i \geq 1$ the first, second and last terms vanish since $A, A' \in \mathfrak{L}$. Hence $\text{Ext}^i(K, B) = 0$ for all $B \in \mathfrak{R}$ and all $i \geq 1$. In particular $K \in \mathfrak{L}$.

The second statement is dual

Now assume that \mathcal{E} has enough \mathfrak{R} -projectives. Note that all projectives are contained in \mathfrak{L} . Let $A \in \mathfrak{L}$. Since \mathcal{E} has enough projectives we may construct an exact sequence

$$0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$$

with P \mathfrak{R} -projective. Since $P \in \mathfrak{L}$, $K \in \mathfrak{L}$ as well. Let $B \in \mathfrak{R}$.

$$\dots \rightarrow \text{Ext}^i(A, B) \rightarrow \text{Ext}^i(P, B) \rightarrow \text{Ext}^i(K, B) \rightarrow \text{Ext}^{i+1}(A, B) \rightarrow \text{Ext}^{i+1}(P, B) \rightarrow \dots$$

Since P is \mathfrak{R} -projective, we get an isomorphism

$$\mathrm{Ext}^i(K, B) \xrightarrow{\sim} \mathrm{Ext}^{i+1}(A, B)$$

and the claim follows by induction.

The last statement is again dual. \square

With this result in hand we can easily generalise [Gil04] Theorem 3.12 to the exact setting.

Theorem 4.39. *Let $(\mathfrak{L}, \mathfrak{R})$ be a hereditary cotorsion pair in an exact category \mathcal{E} . If \mathcal{E} has enough projectives then in $\mathbf{Ch}_*(\mathcal{E})$ for $* \in \{\geq 0, +, \emptyset\}$, $dg\tilde{\mathfrak{R}} \cap \mathfrak{W} = \tilde{\mathfrak{R}}$. If \mathcal{E} has enough injectives then in $\mathbf{Ch}_*(\mathcal{E})$ for $* \in \{\leq 0, -, \emptyset\}$ $dg\tilde{\mathfrak{L}} \cap \mathfrak{W} = \tilde{\mathfrak{L}}$. In particular, if \mathcal{E} has enough projectives and injectives, then the induced cotorsion pairs on \mathcal{E} are compatible.*

Proof. We shall show the first statement, the second being dual.

Let X be an acyclic $dg\mathfrak{R}$ complex. We need to show that $\mathrm{Ext}^1(A, Z_n X) = 0$ for all $A \in \mathfrak{L}$. Let P_\circ be an augmented projective resolution of A

$$P_\circ = \dots \rightarrow P_2 \rightarrow P_1 \rightarrow A \rightarrow 0$$

with P_i projective. Since projectives are in \mathfrak{L} , and the cotorsion pair is hereditary, P_\circ is an \mathfrak{L} -complex.

$$0 \rightarrow Z_n X \rightarrow X_n \rightarrow Z_{n-1} X \rightarrow 0$$

is an exact sequence and $\mathrm{Ext}^1(A, X_n) = 0$ since $X_n \in \mathfrak{R}$. So by the long exact sequence, it is sufficient to show that

$$\mathrm{Hom}(A, X_n) \rightarrow \mathrm{Hom}(A, Z_{n-1} X)$$

is surjective. Let $f : A \rightarrow Z_{n-1} X$ be a morphism. By Lemma 2.51 f induces a chain map $P_\circ[n-1] \rightarrow X$. Since X is a $dg\mathfrak{R}$ complex and $P_\circ[n-1]$ is an \mathfrak{L} -complex this map is homotopic to 0. Any chain homotopy $\{D_n\}$ from f to 0 will give a lift $f = d_n D_{n-1}$. \square

Lemma 3.14 in [Gil04], which partially handles the case in which we may not have enough injectives or projectives also passes essentially unaffected to exact categories.

Lemma 4.40. *Let \mathcal{E} be an exact category and $(\mathfrak{L}, \mathfrak{R})$ a cotorsion pair on \mathcal{E} . Consider the categories $\mathbf{Ch}_*(\mathcal{E})$ for any $* \in \{\geq 0, \leq 0, +, -, b, \emptyset\}$.*

(1) *If $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ is a cotorsion pair with enough projectives and $dg\tilde{\mathfrak{R}} \cap \mathfrak{W} = \tilde{\mathfrak{R}}$ then $dg\tilde{\mathfrak{L}} \cap \mathfrak{W} = \tilde{\mathfrak{L}}$.*

(2) *If $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ is a cotorsion pair with enough injectives and $dg\tilde{\mathfrak{L}} \cap \mathfrak{W} = \tilde{\mathfrak{L}}$ then $dg\tilde{\mathfrak{R}} \cap \mathfrak{W} = \tilde{\mathfrak{R}}$.*

Proof. The statements are dual so we focus on the first.

Let X be an exact $dg\mathfrak{L}$ complex. Since $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ has enough projectives we have a short exact sequence

$$0 \rightarrow B \rightarrow A \rightarrow X \rightarrow 0$$

with $A \in \tilde{\mathfrak{L}}$ and $B \in dg\tilde{\mathfrak{R}}$. Since X and A are each acyclic, so is B by Proposition 4.33. But then B is an \mathfrak{R} -complex by hypothesis. Hence the sequence splits by Proposition 4.30 and X is a direct summand of A . In particular it is an \mathfrak{L} -complex. \square

These next two results partially deal with the issue of completeness.

Lemma 4.41. *Let \mathcal{E} be an exact category. Suppose*

$$0 \longrightarrow B \longrightarrow A \xrightarrow{f} X \longrightarrow 0$$

is a short exact sequence of complexes in the degree wise exact structure with both B and $\mathrm{cone}(f)$ either good or cogood. Then B is acyclic if and only if f is a quasi-isomorphism.

Proof. Let $I : \mathcal{E} \rightarrow \mathcal{A}$ a suitable abelianization. Then by [Wei95] Exercise 1.59 there is a long exact sequence

$$\dots \longrightarrow H_{n+1}(\text{Ker}(I(f_\bullet))) \longrightarrow H_n(\text{cone}(I(f_\bullet))) \longrightarrow H_n(\text{Coker}(I(f_\bullet))) \longrightarrow H_{n-1}(\text{Ker}(I(f))) \longrightarrow \dots$$

If f_\bullet is a quasi-isomorphism, then $\text{cone}(I(f_\bullet))$ is acyclic. It is also an admissible epimorphism, so $\text{Coker}(I(f_\bullet)) = 0$. Hence $\text{Ker}(I(f_\bullet)) = I(B)$ is acyclic.

If B is acyclic then again since $\text{Coker}(I(f_\bullet)) = 0$, $H_n(\text{cone}(I(f_\bullet))) = 0$ as well. Thus $I(f)$ is a quasi-isomorphism, so f is as well. \square

Proposition 4.42. *Let $(\mathfrak{L}, \mathfrak{R})$ be a functorially complete cotorsion pair on an exact category \mathcal{E} . Then the cotorsion pair $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ on both $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ and $\mathbf{Ch}_+(\mathcal{E})$ has enough functorial projectives.*

Proof. Let X_\bullet be an object of $\mathbf{Ch}_*(\mathcal{E})$ where $*$ in $\{\geq 0, +\}$. By an easy adaptation of the proof of Lemma 2.50, one can find a (functorial) quasi-isomorphism $f_\bullet : L_\bullet \rightarrow X_\bullet$ with each L_n an object of \mathfrak{L} , which is an admissible epimorphism, and whose kernel is a complex R_\bullet with $R_n \in \mathfrak{R}$. Now L_\bullet is a $dg\mathfrak{L}$ complex by Lemma 4.29. By Lemma 4.41 R_\bullet is acyclic, and therefore an \mathfrak{R} -complex. So the cotorsion pair has enough (functorial) projectives. \square

This is essentially all that can be said at this level of generality.

4.5.2. Properties of dg -Model Structures.

Definition 4.43. *Let \mathcal{E} be an exact category and $(\mathfrak{L}, \mathfrak{R})$ a cotorsion pair on \mathcal{E} . If $(\tilde{\mathfrak{L}}, dg\tilde{\mathfrak{R}})$ and $(dg\tilde{\mathfrak{L}}, \tilde{\mathfrak{R}})$ are (functorially) complete cotorsion pairs on $\mathbf{Ch}_*(\mathcal{E})$ for $*$ in $\{\geq 0, \leq 0, b, +, -\emptyset\}$ satisfying $dg\mathfrak{L} \cap \mathfrak{W} = \tilde{\mathfrak{L}}$ and $dg\mathfrak{R} \cap \mathfrak{W} = \tilde{\mathfrak{R}}$, then we say $(\mathfrak{L}, \mathfrak{R})$ is dg_* -compatible.*

In particular, if $(\mathfrak{L}, \mathfrak{R})$ is dg_* -compatible, then there is an induced compatible model structure on $\mathbf{Ch}_*(\mathcal{E})$. The resulting model structure will have quasi-isomorphisms as its weak equivalences.

Proposition 4.44. *Suppose that $*$ in $\{\geq 0, \leq 0, +, -, b\}$ cotorsion pair on an exact category \mathcal{E} . The weak equivalences in the induced model structure are precisely the quasi-isomorphisms. If \mathcal{E} has all kernels then this is also true for $*$ in $\{\emptyset\}$.*

Proof. First we show that admissible monics and admissible epics which are weak equivalences are quasi-isomorphisms. We will show it for monics, the case of epics being dual. Let $f : A \rightarrow B$ be an admissible monic which is a weak equivalence. It is sufficient to show that $I(f)$ is quasi-isomorphism, where $I : \mathcal{E} \rightarrow \mathcal{A}(\mathcal{E})$ is a suitable abelianization. Now we have an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

with $C \in \mathfrak{W}$. In particular, C is acyclic. By (the dual of) Lemma 4.41, f is a quasi-isomorphism.

Let f be a morphism of $\mathbf{Ch}_*(\mathcal{E})$. Factor it as $p \circ i$ where i is a fibration, p is a cofibration and either p or i is trivial, and therefore a quasi-isomorphism. By the exact triangle

$$\text{cone}(i) \rightarrow \text{cone}(f) \rightarrow \text{cone}(p) \rightarrow^{+1}$$

and the fact that acyclic complexes form a thick subcategory, we find that f is a quasi-isomorphism if and only the other factor is trivial. \square

Remark 4.45. *The previous result says that the homotopy category of a model structure arising from a dg -compatible cotorsion pair is the derived category (for $*$ in $\{+, -, b, \emptyset\}$).*

Such model structures are also both left and right proper. More generally, we have the following.

Proposition 4.46. *Let \mathcal{E} be an exact category. Let $*$ in $\{\geq 0, \leq 0, +, -, b\}$. Suppose there is a model structure on $\mathbf{Ch}_*(\mathcal{E})$ whose weak equivalences are the quasi-isomorphisms and such that any cofibration is an admissible monomorphism in each degree. Then the model structure is left proper. If \mathcal{E} has all kernels then this is also true for $\mathbf{Ch}(\mathcal{E})$. Dually, if any fibration is an admissible epimorphism in each degree then the model structure is right proper.*

Proof. The dual case is slightly easier to write down, so we will prove that. We need to check that, given a pull-back diagram

$$\begin{array}{ccc} A_{\bullet} & \xrightarrow{p'} & B_{\bullet} \\ \downarrow q' & & \downarrow q \\ X_{\bullet} & \xrightarrow{p} & Y_{\bullet} \end{array}$$

where p is an admissible epic, and q is a quasi-isomorphism, then q' is a quasi-isomorphism. By Lemma 2.8 without loss of generality, we may assume that the category \mathcal{E} is actually abelian. We argue by elements. A_{\bullet} is isomorphic to

$$\{(x, b) \in X_{\bullet} \times B_{\bullet} : p(x) = q(b)\}$$

with q' and p' being the restrictions of the projections. Suppose $(x, b) \in \text{Ker} d_n^A$ is such that $q'(x, b) = x = 0$. But then $q(b) = p(x) = 0$. So $b = d_{n+1}^B(\tilde{b})$ for some b , and $(x, b) = d_{n+1}^A((0, \tilde{b}))$.

Now suppose $x \in \text{Ker} d_n^X$. Then $p(x) \in \text{Ker} d_n^Y$. Thus there is a $b \in \text{Ker} d_n^B$ and a $\tilde{y} \in Y_{n+1}$ such that $q(b) = p(x) + d_{n+1}^Y(\tilde{y})$. Now, p is an epic, so there is $\tilde{x} \in X_{n+1}$ such that $\tilde{y} = p(\tilde{x})$. Write $a = (x + d_{n+1}^X(\tilde{x}), b)$. Then $a \in A_{\bullet}$ and $q'(a) = x + d_{n+1}^X(\tilde{x})$. This shows that q' is a quasi-isomorphism. \square

4.5.3. Small dg-Cotorsion Pairs. Let us now examine when the cotorsion pair $(\tilde{\mathcal{L}}, dg\tilde{\mathcal{R}})$ is small. The following result is adapted from [Gil07] Proposition 3.8.

Proposition 4.47. *Let $(\mathcal{L}, \mathcal{R})$ be a cotorsion pair in an exact category \mathcal{E} which has a set of admissible generators \mathcal{G} . Suppose that $(\mathcal{L}, \mathcal{R})$ is cogenerated by a set $\{A_i\}_{i \in I}$. Then $(dg\tilde{\mathcal{L}}, \tilde{\mathcal{R}})$ is cogenerated by the set*

$$S = \{S^n(G) : G \in \mathcal{G}, n \in \mathbb{Z}\} \cup \{S^n(A_i) : n \in \mathbb{Z}, i \in I\}$$

for $* \in \{+\}$ (and. $* \in \{\emptyset\}$ if \mathcal{E} has kernels) and

$$S = \{S^n(G) : G \in \mathcal{G}, n \geq 0\} \cup \{S^n(A_i) : n \geq 0, i \in I\}$$

for $* \in \{\geq 0\}$.

Furthermore, suppose $(\mathcal{L}, \mathcal{R})$ is small with generating morphisms the map $\{0 \rightarrow G : G \in \mathcal{G}\}$ together with monics k_i as below (one for each $i \in I$):

$$0 \longrightarrow Y_i \xrightarrow{k_i} Z_i \longrightarrow A_i \longrightarrow 0$$

Then $(dg\tilde{\mathcal{L}}, \tilde{\mathcal{R}})$ is small with generating morphisms the set

$$\tilde{I} = \{0 \rightarrow D^n(G)\} \cup \{S^{n-1}(G) \rightarrow D^n(G)\} \cup \{S^n(k_i) : S^n(Y_i) \rightarrow S^n(Z_i)\}$$

for $* \in \{+\}$ (and. $* \in \{\emptyset\}$ if \mathcal{E} has kernels) and

$$\tilde{I} = \{0 \rightarrow S^0(G)\} \cup \{0 \rightarrow D^n(G) : n > 0\} \cup \{S^{n-1}(G) \rightarrow D^n(G) : n > 0\} \cup \{S^n(k_i) : S^n(Y_i) \rightarrow S^n(Z_i) : n \geq 0\}$$

for $* \in \{\geq 0\}$.

Proof. We prove the result for $* \in \{+, \emptyset\}$ and then explain how to modify it for $* \in \{\geq 0\}$.

By Proposition 4.29, $S \subseteq dg\tilde{\mathcal{L}}$. Hence $\tilde{\mathcal{R}} \subset S^{\perp}$.

Conversely suppose $X \in S^{\perp}$. We first show that X is acyclic. Consider the short exact sequences

$$0 \rightarrow S^{n-1}(G) \rightarrow D^n(G) \rightarrow S^n(G) \rightarrow 0$$

We get an exact sequence of abelian groups

$$\text{Hom}(D^n(G), X) \rightarrow \text{Hom}(S^{n-1}(G), X) \rightarrow \text{Ext}^1(S^n(G), X)$$

But

$$\text{Ext}^1(S^n(G), X) = 0$$

by assumption. Hence $\text{Hom}(D^n(G), X) \rightarrow \text{Hom}(S^{n-1}(G), X)$ is surjective for each G and X is acyclic. Now for all $i \in I$ we have by Lemma 3.10

$$0 = \text{Ext}_{\mathbf{Ch}(\mathcal{E})}^1(S^n(A_i), X) \cong \text{Ext}_{\mathcal{E}}^1(A_i, Z_n X)$$

Since $\{A_i\}$ cogenerates the cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ we get $Z_n X \in \mathfrak{R}$.

Finally we prove smallness. By Proposition 3.15, the set $\{D^n(G) : G \in \mathcal{G}\}$ generates $\mathbf{Ch}_*(\mathcal{E})$. By Proposition 4.29, $D^n(G) \in dg\tilde{\mathfrak{L}}$. In particular $dg\tilde{\mathfrak{L}}$ contains these generators. Let X be any chain complex. If for each G the map

$$\text{Hom}(D^n(G), X) \rightarrow \text{Hom}(S^{n-1}(G), X)$$

is a surjection, then X is acyclic. It remains to show that $Z_n X \in \mathfrak{R}$. By Lemma 3.10, a map $Y_i \rightarrow Z_n X$ induces a morphism $S^n(Y_i) \rightarrow X$. By assumption this extends over $S^n(k_i)$ to a map $S^n(Z_i) \rightarrow X$. Again by Lemma 3.10 this implies that any map $Y_i \rightarrow Z_n X$ extends over k_i to a map $Z_i \rightarrow Z_n X$. By assumption this means $Z_n X \in \mathfrak{R}$.

Now consider the case $* \in \{\geq 0\}$. The only difference in the proof is that now the generating set for $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ is $\{D^n(G) : G \in \mathcal{G} : n > 0\} \cup \{S^0(G) : G \in \mathcal{G}\}$. This is also a subset of $dg\tilde{\mathfrak{L}}$. \square

Remark 4.48. *In the situation of the previous proposition, if the domains of the generating morphisms for the cotorsion pair $(\mathfrak{L}, \mathfrak{R})$ are tiny, then the domains of the maps in I are also tiny by Proposition 3.16.*

5. THE PROJECTIVE MODEL STRUCTURE AND THE DOLD-KAN CORRESPONDENCE

5.1. The Projective Model Structure. In this section \mathcal{E} is an exact category with enough functorial projectives. We denote the collection of all projective objects in \mathcal{E} by $\mathbf{Proj}(\mathcal{E})$

Definition 5.1. *Let \mathcal{E} be an exact category. If it exists, the **projective model structure** on $\mathbf{Ch}_*(\mathcal{E})$, for $* \in \{+, \emptyset\}$ is the model structure in which*

- *Weak equivalences are quasi-isomorphisms.*
- *Fibrations are degree-wise admissible epics.*
- *Cofibrations are maps which have the left-lifting property with respect to acyclic fibrations.*

Proposition 5.2. *Let \mathcal{E} be an exact category. Suppose that the cotorsion pair $(dg\tilde{\mathbf{Proj}}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$ on $\mathbf{Ch}_*(\mathcal{E})$ for $* \in \{+, \geq 0, \emptyset\}$ has enough functorial projectives. Then it has enough functorial injectives.*

Proof. Let X_\bullet be an object of $\mathbf{Ch}_*(\mathcal{E})$, and let $f_\bullet : L_\bullet \rightarrow X_\bullet$ be a quasi-isomorphism and admissible epimorphism with acyclic kernel, and $L_\bullet \in dg\tilde{\mathbf{Proj}}(\mathcal{E})$.

We have a short exact sequence

$$0 \rightarrow X_\bullet \rightarrow \text{cone}(f_\bullet) \rightarrow L_\bullet[-1] \rightarrow 0$$

$\text{cone}(f_\bullet)$ is an acyclic complex, so it is in $\mathbf{Ob}(\mathcal{E})$. Clearly $L_\bullet[-1] \in dg\tilde{\mathbf{Proj}}(\mathcal{E})$. \square

We are now ready to prove the following theorem.

Theorem 5.3. *Let \mathcal{E} be an exact category with enough projectives. Then the projective model structure exists on $\mathbf{Ch}_+(\mathcal{E})$ and is compatible. It is functorial if \mathcal{E} has enough functorial projectives. It is cellular if \mathcal{E} is elementary, and combinatorial if \mathcal{E} is locally presentable. If \mathcal{E} has all kernels and the functor $\lim_{\rightarrow \mathbb{N}}$ exists and is exact then this is all true for $\mathbf{Ch}(\mathcal{E})$ as well.*

Proof. Consider the projective cotorsion pair $(\mathbf{Proj}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$ on \mathcal{E} . By Corollary 4.32, $(dg\tilde{\mathbf{Proj}}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$ is a cotorsion pair on $\tilde{\mathbf{Ch}}_+(\mathcal{E})$. It is functorially complete by Proposition 4.42 and Proposition 5.2

We claim that $(\mathbf{Proj}(\mathcal{E}), dg\tilde{\mathbf{Ob}}(\mathcal{E}))$ is also a cotorsion pair on $\mathbf{Ch}_+(\mathbf{Ob}(\mathcal{E}))$. First note that $\mathbf{Proj}(\mathcal{E})$ consists of split exact complexes of projectives. By Proposition 3.13 this is precisely the class of projective objects in $\mathbf{Ch}_+(\mathcal{E})$. Then by Proposition 5.11 $dg\tilde{\mathbf{Ob}}(\mathcal{E}) = \mathbf{Ch}_+(\mathbf{Ob}(\mathcal{E}))$. Hence $(\mathbf{Proj}(\mathcal{E}), dg\tilde{\mathbf{Ob}}(\mathcal{E}))$ is just

the projective cotorsion pair. Now $\mathbf{Ob}(\mathcal{E})$ is the class of all acyclic complexes, \mathfrak{W} . Thus $dg\mathbf{Ob}(\mathcal{E}) \cap \mathfrak{W} = \mathbf{Ch}_+(\mathcal{E}) \cap \mathfrak{W} = \mathfrak{W} = \mathbf{Ob}(\mathcal{E})$. Moreover $\mathbf{Ch}_+(\mathcal{E})$ has enough projectives by Corollary 3.14. By Lemma 4.40 it remains to prove that $(\mathbf{Proj}(\mathcal{E}), dg\mathbf{Ob}(\mathcal{E}))$ is (functorially) complete. But in a category with enough (functorial) projectives the projective cotorsion pair is always (functorially) complete by Example 4.4.

Assume further that \mathcal{E} is elementary. Then by Example 4.15, the cotorsion pair $(\mathbf{Proj}(\mathcal{E}), dg\mathbf{Ob}(\mathcal{E}))$ is small and by Proposition 4.47 the cotorsion pair $(dg\mathbf{Proj}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$ is small. By Lemma 4.16, the model structure is cellular. The fact about combinatoriality is clear.

The proof for unbounded complexes works in almost exactly the same way. All that needs to be verified in this case is that $(dg\mathbf{Proj}(\mathcal{E}), \mathbf{Ob}(\mathcal{E}))$ is complete. Now the class of projectives is closed under \mathbb{N} -indexed extensions by Lemma 4.1. Completeness therefore follows from Corollary 2.56, Proposition 2.57 and Proposition 5.2. \square

Remark 5.4. *The existence of the projective model structure on bounded below chain complexes on a quasi-abelian category with enough projectives was already known to Bühler [Büh11] (see Appendix C). The proof there is more direct. In fact the proof works for any idempotent complete exact category in which the class of all kernel-cokernel pairs forms the exact structure (all kernels and cokernels need not exist).*

Recall that if \mathcal{E} is (quasi-)elementary quasi-abelian, then Proposition 3.7 says that $\mathcal{LH}(\mathcal{E})$ is as well. Thus the projective model structure exists on $\mathbf{Ch}(\mathcal{LH}(\mathcal{E}))$. Moreover the induced functor $I : \mathbf{Ch}(\mathcal{E}) \rightarrow \mathbf{Ch}(\mathcal{LH}(\mathcal{E}))$ is then right Quillen. Indeed it is left adjoint to the induced functor $C : \mathbf{Ch}\mathcal{LH}(\mathcal{E}) \rightarrow \mathbf{Ch}(\mathcal{E})$. It preserves fibrations since $I : \mathcal{E} \rightarrow \mathcal{LH}(\mathcal{E})$ is a left abelianization, and it preserves quasi-isomorphisms by Corollary 2.84. Moreover by Theorem 2.82, Proposition 2.83 and Proposition 4.44 it induces an equivalence between the homotopy categories. We therefore have

Proposition 5.5. *Let \mathcal{E} be an elementary quasi-abelian category. Then the adjunction*

$$\begin{array}{ccc} & \xrightarrow{C} & \\ \mathbf{Ch}(\mathcal{LH}(\mathcal{E})) & & \mathbf{Ch}(\mathcal{E}) \\ & \xleftarrow{I} & \end{array}$$

is a Quillen equivalence between the projective model structures.

We claim that the projective model structure exists also on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ for \mathcal{E} an exact category with kernels. It will be left pseudo-compatible, but not compatible.

Definition 5.6. *Let \mathcal{E} be an exact category. If it exists, the **projective model structure** on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$, is the model structure in which*

- *Weak equivalences are quasi-isomorphisms.*
- *Fibrations are degree-wise admissible epics in each strictly positive degree.*
- *Cofibrations are maps which have the left-lifting property with respect to acyclic fibrations.*

Theorem 5.7. *Let \mathcal{E} be an exact category with enough projectives and which has all kernels. Then the projective model structure exists on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$. Moreover it is a left pseudo-compatible model structure with Waldhausen pair $(dg\mathbf{Proj}(\mathcal{E}), \mathfrak{W})$. In particular the acyclic cofibrations are the degree-wise admissible monics whose cokernels are split exact complexes of projectives. If \mathcal{E} is elementary then it is cellular. In particular if \mathcal{E} is locally presentable and elementary then the projective model structure is combinatorial.*

Proof. The class of weak equivalences satisfies the 2-out-of-6 property since it does so in $\mathbf{Ch}_+(\mathcal{E})$. Denote the class of fibrations by \mathcal{F} and of weak equivalences by \mathcal{W} . Also denote the class of admissible monomorphisms with degree-wise projective cokernel by \mathcal{C} . Let us show that $\mathcal{F} \cap \mathcal{W}$ consists of quasi-isomorphisms which are admissible epimorphisms in each degree. In order to do this, one may first pass to a left abelianization and assume that \mathcal{E} is abelian. Here the argument is a simple diagram chase. By Proposition 4.42 and Proposition

5.2, it follows $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is a (compatible) weak factorisation system with corresponding cotorsion pair $(dg\mathbf{Proj}(\mathcal{E}), \mathfrak{W})$. In particular the cofibrations in the sense of Definition 5.6 coincide with the class \mathcal{C} . It therefore remains to check that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system.

Let us first check the lifting conditions. First suppose a map $A_\bullet \rightarrow B_\bullet$ in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ has the left lifting property with respect to maps $X_\bullet \rightarrow Y_\bullet$ in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ which are admissible epimorphisms in each strictly positive degree. Let $E_\bullet \rightarrow F_\bullet$ be a map between any complexes in $\mathbf{Ch}(\mathcal{E})$ which is an admissible epimorphism in all degrees. Consider a diagram

$$\begin{array}{ccc} A_\bullet & \longrightarrow & E_\bullet \\ \downarrow & & \downarrow \\ B_\bullet & \longrightarrow & F_\bullet \end{array}$$

Since A_\bullet and B_\bullet are in $\mathbf{Ch}_{\geq 0}$ we can factor the above diagram as

$$\begin{array}{ccccc} A_\bullet & \longrightarrow & \tau_{\geq 0} E_\bullet & \longrightarrow & E_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ B_\bullet & \longrightarrow & \tau_{\geq 0} F_\bullet & \longrightarrow & F_\bullet \end{array}$$

Now the map $\tau_{\geq 0} E_\bullet \rightarrow \tau_{\geq 0} F_\bullet$ is an epimorphism in each strictly positive degree. By assumption we can find a lift as follows.

$$\begin{array}{ccccc} A_\bullet & \longrightarrow & \tau_{\geq 0} E_\bullet & \longrightarrow & E_\bullet \\ \downarrow & \nearrow & \downarrow & & \downarrow \\ B_\bullet & \longrightarrow & \tau_{\geq 0} F_\bullet & \longrightarrow & F_\bullet \end{array}$$

Thus the map $A_\bullet \rightarrow B_\bullet$ has the left-lifting property with respect to all degree-wise epimorphisms in $\mathbf{Ch}_+(\mathcal{E})$. By Theorem 5.3 $A_\bullet \rightarrow B_\bullet$ is an admissible monic whose cokernel is a split exact complex of projectives. Now, any acyclic cofibration is of the form $A_\bullet \rightarrow A_\bullet \oplus \left(\bigoplus_{n>0} D^n(P_n) \right)$ where each P_n is a projective object in \mathcal{E} , and the map is the inclusion into the first factor of the direct sum. Clearly then it is enough to show that the collection of maps $\{0 \rightarrow D^n(P) : n > 0, P \text{ is projective}\}$ has the left lifting property with respect to \mathcal{F} , and that a map is in \mathcal{F} if and only if it has the right-lifting property with respect to these maps. However this follows from Lemma 3.10 and Proposition 3.2.

It remains to find a (functorial) factorisation. Let $f_\bullet : X_\bullet \rightarrow Y_\bullet$ be a map in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$. We can factor it in $\mathbf{Ch}_+(\mathcal{E})$ as

$$X_\bullet \rightarrow X_\bullet \oplus \left(\bigoplus_{n \geq 0} D^n(P_n) \right) \rightarrow Y_\bullet$$

where $X_\bullet \rightarrow X_\bullet \oplus \left(\bigoplus_{n \geq 0} D^n(P_n) \right)$ is the inclusion into the first factor, and $X_\bullet \oplus \left(\bigoplus_{n \geq 0} D^n(P_n) \right) \rightarrow Y_\bullet$ is an admissible epimorphism in each degree. Then

$$X_\bullet \rightarrow X_\bullet \oplus \left(\bigoplus_{n > 0} D^n(P_n) \right) \rightarrow Y_\bullet$$

is also a factorisation of f_\bullet , $X_\bullet \rightarrow X_\bullet \oplus \left(\bigoplus_{n > 0} D^n(P_n) \right)$ is an acyclic cofibration in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$, and $X_\bullet \oplus \left(\bigoplus_{n > 0} D^n(P_n) \right) \rightarrow Y_\bullet$ is an admissible epimorphism in each strictly positive degree.

We prove the statement about cellularity. Suppose that \mathcal{P} is a projective generating set consisting of tiny objects. It follows from Proposition 4.47 that the weak factorisation system $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ is cellular. From our proof above that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ is a weak factorisation system, it follows that $\{0 \rightarrow D^n(P) : n > 0, P \in \mathcal{P}\}$ is a set of generating morphisms for $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, so it is also a cellular weak factorisation system. The claim about combinatoriality is clear. \square

Remark 5.8. *The existence of the projective model structure on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ in the case that \mathcal{E} is quasi-abelian was also known. This is mentioned in a math.stackexchange.com exchange, [uh], as an adaptation of the proof for $\mathbf{Ch}_+(\mathcal{E})$ in [Büh11].*

5.2. The Projective Model Structure on Monoidal Exact Categories. We now turn our attention to monoidal model structures on categories of chain complexes.

Proposition 5.9. *Let $(\mathcal{E}, \otimes, k)$ be an additive symmetric monoidal category with \mathcal{E} an exact category. For $*$ $\in \{\geq 0, \leq 0, b, +, -\}$ the flat objects in $(\mathbf{Ch}_*(\mathcal{E}), \otimes, S^0(k))$ are precisely the complexes F_\bullet in $\mathbf{Ch}_*(\mathcal{E})$ such that for each $n \in \mathbb{N}$, F_n is flat. If in addition countable direct sums exist and are exact, then the flat objects in $(\mathbf{Ch}(\mathcal{E}), \otimes, S^0(k))$ are also the complexes F_\bullet that for each $n \in \mathbb{N}$, F_n is flat*

Proof. Let

$$0 \longrightarrow X_\bullet \longrightarrow Y_\bullet \longrightarrow Z_\bullet \longrightarrow 0$$

be a short exact sequence in $\mathbf{Ch}_*(\mathcal{E})$. Let F_\bullet be a complex. Then the n th row of

$$0 \longrightarrow X_\bullet \otimes F_\bullet \longrightarrow Y_\bullet \otimes F_\bullet \longrightarrow Z_\bullet \otimes F_\bullet \longrightarrow 0$$

is

$$0 \longrightarrow \bigoplus_{i+j=n} X_i \otimes F_j \longrightarrow \bigoplus_{i+j=n} Y_i \otimes F_j \longrightarrow \bigoplus_{i+j=n} Z_i \otimes F_j \longrightarrow 0$$

Since the direct sums involved are exact, this sequence is short exact if for each i, j ,

$$0 \longrightarrow X_i \otimes F_j \longrightarrow Y_i \otimes F_j \longrightarrow Z_i \otimes F_j \longrightarrow 0$$

is short exact. It follows immediately that a complex whose entries are flat in \mathcal{E} is itself a flat object in $\mathbf{Ch}_*(\mathcal{E})$. To see that a flat complex must have flat entries, simply take a short exact sequence in \mathcal{E} , and regard it as a short exact sequence in $\mathbf{Ch}_*(\mathcal{E})$ concentrated in degree 0. \square

We are going to use Theorem 4.18. In order to deal with the third condition of that theorem we are going to need the following notion.

Definition 5.10. *An acyclic complex $F_\bullet \in \mathbf{Ch}(\mathcal{E})$ is said to be \otimes -stably acyclic if for any complex X_\bullet , $F_\bullet \otimes X_\bullet$ is acyclic. An acyclic complex $F_\bullet \in \mathbf{Ch}(\mathcal{E})$ is said to be **Hom-stably acyclic** if for any complex X_\bullet , $\mathbf{Hom}(F_\bullet, X_\bullet)$ is acyclic.*

Proposition 5.11. *Let F be a flat object in \mathcal{E} . Then for all n , the complex $D^n(F)$ is \otimes -stably acyclic. In particular, split exact complexes of flat objects are \otimes -stably acyclic. If F is projective then $D^n(F)$ is **Hom-stably acyclic**. Hence split exact complexes of projectives are **Hom-stably acyclic**.*

Proof. Clearly it is sufficient to prove the proposition for $n = 1$. In this case, $D^1(F) \otimes X_\bullet \cong F \otimes \text{cone}(id_{X_\bullet})$. Since F is flat this complex is acyclic. The proof for projectives is similar. \square

We would also like our model structure to satisfy the monoid axiom. Towards this we note the following.

Proposition 5.12. *Let \mathcal{E} be a cocomplete elementary exact category. Then transfinite compositions of quasi-isomorphisms in $\mathbf{Ch}(\mathcal{E})$ are quasi-isomorphisms.*

Proof. The proof is by transfinite induction. Since a finite composition of quasi-isomorphisms is a quasi-isomorphism, the successor part of the induction is finished. Now let λ be a limit ordinal and $F : \lambda \rightarrow \mathbf{Ch}(\mathcal{E})$ a continuous functor. For $\alpha \leq \beta \leq \lambda$ denote by $f_{\alpha, \beta}$ the map $F_\alpha \rightarrow F_\beta$. For $\beta \leq \lambda$ write $f_\beta = f_{0, \beta}$. It is clear that

$$\text{cone}(f_\lambda) \cong \lim_{\beta < \lambda} \text{cone}(f_\beta)$$

Since each f_β is a quasi-isomorphism, $\text{cone}(f_\beta)$ is acyclic. Since \mathcal{E} is elementary, this implies $\lim_{\beta < \lambda} \text{cone}(f_\beta)$ is acyclic, which would mean that $\text{cone}(f_\lambda)$ is acyclic and hence that f_λ is a quasi-isomorphism. \square

We are now ready to prove the following

Theorem 5.13. *Let \mathcal{E} be a projectively monoidal exact category with enough projectives. Then the projective model structure on $\mathbf{Ch}_+(\mathcal{E})$ is monoidal. If \mathcal{E} also has kernels, then the projective model structure on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ is monoidal. If in addition \mathcal{E} is elementary then $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ satisfies the monoid axiom.*

Proof. By Proposition 5.9 the cofibrant objects are flat. By Lemma 4.22, all cofibrations are pure. Now since $\mathbf{Proj}(\mathcal{E})$ is closed under \otimes , $dg\tilde{\mathbf{Proj}}(\mathcal{E})$ is closed under \otimes by Lemma 4.29. Now let L, L' be $dg\tilde{\mathbf{Proj}}(\mathcal{E})$ -complexes with L' acyclic. We have to show that $L \otimes L'$ is acyclic. However any complex in $\mathbf{Proj}(\mathcal{E})$ is a split exact complex of objects in $\mathbf{Proj}(\mathcal{E})$. By Proposition 5.11 such an object is \otimes -stably acyclic.

For the assertions about the monoid axiom, we must check the conditions of Theorem 4.24. The first condition is guaranteed, again because the trivially cofibrant objects are \otimes -stably acyclic. The second condition follows from Proposition 5.12 and by Proposition 4.44. \square

We would like a version for unbounded complexes. We are going to prove the following.

Theorem 5.14. *Let $(\mathcal{E}, \otimes, k)$ be a bicomplete closed, monoidal elementary exact category such that arbitrary product functors are admissibly exact and arbitrary coproduct functors are admissibly coexact. Then $(\mathbf{Ch}(\mathcal{E}), \otimes, \underline{\mathbf{Hom}}, S^0(k))$ is a monoidal model category which satisfies the monoid axiom.*

Proof. The condition on products and coproducts guarantee that $(\mathbf{Ch}(\mathcal{E}), \otimes, \underline{\mathbf{Hom}}, S^0(k))$ is actually a monoidal exact category. Now the proof goes through in almost exactly the same way as Proposition 5.13. All that remains to prove is that the tensor product of two complexes in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$ is again in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$. This is a consequence of the proposition below, and its corollary. \square

Proposition 5.15. *If X_\bullet is in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$ then for any acyclic complex Y_\bullet , $\underline{\mathbf{Hom}}(X_\bullet, Y_\bullet)$ is acyclic.*

Proof. Let \mathcal{P} be a generating set of projectives. Then by Observation 2.71, for each $P \in \mathcal{P}$, $\underline{\mathbf{Hom}}(S^0(P), Y_\bullet)$ is acyclic. Moreover $S^0(P) \in dg\tilde{\mathbf{Proj}}(\mathcal{E})$, so $\underline{\mathbf{Hom}}(S^0(P), Y_\bullet)$ is acyclic by assumption. Now

$$\begin{aligned} \mathbf{Hom}(S^0(P), \underline{\mathbf{Hom}}(X_\bullet, Y_\bullet)) &\cong \mathbf{Hom}(S^0(P) \otimes X_\bullet, Y_\bullet) \\ &\cong \mathbf{Hom}(X_\bullet, \underline{\mathbf{Hom}}(S^0(P), Y_\bullet)) \end{aligned}$$

Since X_\bullet is in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$, $\mathbf{Hom}(X_\bullet, \underline{\mathbf{Hom}}(S^0(P), Y_\bullet))$ is acyclic. Since \mathcal{P} is a projective generating set in a quasi-abelian category $\underline{\mathbf{Hom}}(X_\bullet, Y_\bullet)$ is acyclic. \square

Corollary 5.16. *If X_\bullet and Z_\bullet in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$, so is $X_\bullet \otimes Z_\bullet$.*

Proof. Since $\mathbf{Proj}(\mathcal{E})$ is closed under countable direct sums the entries of the tensor product are objects in $\mathbf{Proj}(\mathcal{E})$. Let Y_\bullet be an acyclic complex.

$$\mathbf{Hom}(X_\bullet \otimes Z_\bullet, Y_\bullet) \cong \mathbf{Hom}(X_\bullet, \underline{\mathbf{Hom}}(Z_\bullet, Y_\bullet))$$

By the Proposition, $\underline{\mathbf{Hom}}(Z_\bullet, Y_\bullet)$ is acyclic. Since X_\bullet is in $dg\tilde{\mathbf{Proj}}(\mathcal{E})$, $\mathbf{Hom}(X_\bullet, \underline{\mathbf{Hom}}(Z_\bullet, Y_\bullet))$ is acyclic. \square

5.3. The Dold-Kan Correspondence. In this section we generalise the Dold-Kan correspondence for abelian groups to elementary exact categories. If \mathcal{C} is a category, we denote by $s\mathcal{C}$ the functor category $[\Delta^{op}, \mathcal{C}]$, where Δ is the usual simplicial category.

Let us recall the Dold-Kan correspondence for abelian categories. The exposition here follows [Wei95] 8.4. For an abelian category \mathcal{A} , there are functors

$$\Gamma : \mathbf{Ch}_{\geq 0}(\mathcal{A}) \rightarrow s\mathcal{A}$$

and

$$N : s\mathcal{A} \rightarrow \mathbf{Ch}_{\geq 0}(\mathcal{A})$$

constructed as follows:

Given an object $A \in s\mathcal{A}$ set

$$NA_n = \bigcap_{i=0}^{n-1} \text{Ker}(d_i)$$

Define a differential $\delta_n = (-1)^n d_n : NA_n \rightarrow NA_{n-1}$. It follows from the simplicial relations that NA_\bullet is a chain complex. Moreover, since by definition a map of simplicial objects commutes with the face maps, this construction is functorial.

The construction of Γ is more involved. For a chain complex $C \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, one sets

$$\Gamma(C)_n = \bigoplus_{\eta: [n] \twoheadrightarrow [p], p \leq n} C_\eta$$

where for $\eta : [n] \twoheadrightarrow [p]$, $C_\eta = C_p$. Given a morphism $\alpha : [n] \rightarrow [m]$ in Δ , define a morphism $\Gamma(C)(\alpha) : \Gamma_m(C) \rightarrow \Gamma_n(C)$ by its restriction $\Gamma(\alpha, \eta) : C_\eta \rightarrow \Gamma(C)$ to each summand C_η as follows. For each surjection $\eta : [n] \twoheadrightarrow [p]$ we consider its epi-mono factorisation $\epsilon\eta'$ of $\eta\alpha$.

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \downarrow \eta' & & \downarrow \eta \\ [q] & \xrightarrow{\epsilon} & [p] \end{array}$$

If $p = q$ so that $\eta\alpha = \eta'$ then we take $\Gamma(\alpha, \eta)$ to be the natural identification of C_η with the summand $C_{\eta'}$ of Γ_m . If $p = q + 1$ and $\epsilon = \epsilon_p$, so that the image of $\eta\alpha$ is $\{0, \dots, p-1\}$, then we take $\Gamma(\alpha, \eta)$ to be the composition

$$C_\eta = C_p \xrightarrow{d} C_{p-1} = C_{\eta'} \longrightarrow \Gamma_m(C)$$

Otherwise we take $\Gamma(\alpha, \eta)$ to be 0.

The Dold-Kan Correspondence says the following

Theorem 5.17 (Dold-Kan for Abelian Categories). *Let \mathcal{A} be an abelian category. Then the functors*

$$\Gamma : \mathbf{Ch}_{\geq 0}(\mathcal{A}) \rightarrow \mathbf{s}\mathcal{A}$$

and

$$N : \mathbf{s}\mathcal{A} \rightarrow \mathbf{Ch}_{\geq 0}(\mathcal{A})$$

form an equivalence of categories.

Proof. See [Wei95] §8.4. □

The constructions of Γ and N make sense in any exact category which has kernels. Thus for an exact category \mathcal{E} with kernels we get functors

$$\Gamma : \mathbf{Ch}_{\geq 0}(\mathcal{E}) \rightarrow \mathbf{s}\mathcal{E}$$

and

$$N : \mathbf{s}\mathcal{E} \rightarrow \mathbf{Ch}_{\geq 0}(\mathcal{E})$$

constructed mutatis mutandis as above.

Corollary 5.18 (Dold-Kan for Exact Categories). *Let \mathcal{E} be an exact category with kernels. Then the functors*

$$\Gamma : \mathbf{Ch}_{\geq 0}(\mathcal{E}) \rightarrow \mathbf{s}\mathcal{E}$$

and

$$N : \mathbf{s}\mathcal{E} \rightarrow \mathbf{Ch}_{\geq 0}(\mathcal{E})$$

form an equivalence of categories.

Proof. Pick a left abelianization $I : \mathcal{E} \rightarrow \mathcal{A}$. Then I extends to functors $\mathbf{s}\mathcal{E} \rightarrow \mathbf{s}\mathcal{A}$ and $\mathbf{Ch}_{\geq 0}(\mathcal{E}) \rightarrow \mathbf{Ch}_{\geq 0}(\mathcal{A})$, which we will also denote by I . Since I preserves kernels we get a commutative diagram.

$$\begin{array}{ccc} \mathbf{s}\mathcal{A} & \xrightarrow{N} & \mathbf{Ch}_{\geq 0}(\mathcal{A}) \\ \uparrow I & & \uparrow I \\ \mathbf{s}\mathcal{E} & \xrightarrow{N} & \mathbf{Ch}_{\geq 0}(\mathcal{E}) \end{array}$$

It is also clear from the construction of Γ that the following diagram commutes

$$\begin{array}{ccc} \mathbf{sA} & \xleftarrow{\Gamma} & \mathbf{Ch}_{\geq 0}(\mathcal{A}) \\ \uparrow I & & \uparrow I \\ \mathbf{sE} & \xleftarrow{\Gamma} & \mathbf{Ch}_{\geq 0}(\mathcal{E}) \end{array}$$

Since the functor I is fully faithful, Theorem 5.17 implies the result. \square

If $\mathcal{A} = \mathbf{Ab}$ is just the category of abelian groups, then there is a well-known model structure on the category \mathbf{sAb} . The weak equivalences (resp. fibrations) are those maps of simplicial abelian groups which are weak equivalences (resp. fibrations) on the underlying simplicial set. As usual, the cofibrations are maps of simplicial abelian groups which have the left-lifting property with respect to the trivial fibrations. Moreover, the category \mathbf{Ab} is an elementary abelian category. As a set of tiny projective generators we can take $\mathcal{P} = \{\mathbb{Z}\}$. Thus there is a projective model structure on $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$. In this case the functors N and Γ also form a Quillen equivalence between these model categories. For a proof see [GJ09] Chapter 3 Section 2. The model structure on \mathbf{sAb} is a special case of a much more general model structure.

Notation 5.19. (1) Let Z be an object in a category \mathcal{C} . We denote by $\mathbf{s}Z$ the constant simplicial object in \mathbf{sC} which is Z in each degree, and such that the face and degeneracy maps are all id_Z .

(2) If \mathcal{C} is additive, then the category \mathbf{sC} is enriched over \mathbf{sAb} in an obvious way. We denote the enriched hom functor by $\mathbf{Hom}_{\mathbf{sC}}$

Theorem 5.20. Suppose that \mathcal{C} is a small bicomplete category, and let $Z = \{Z_i : i \in I\}$ be a set of compact objects of \mathcal{C} . Then \mathbf{sC} is a simplicial model category with $A \rightarrow B$ a weak equivalence (respectively fibration) if and only if the induced map

$$\mathbf{Hom}_{\mathbf{sC}}(\mathbf{s}Z_i, A) \rightarrow \mathbf{Hom}_{\mathbf{sC}}(\mathbf{s}Z_i, B)$$

is a weak equivalence (respectively fibration) for all $i \in I$.

Proof. See [GJ09] Theorem 6.9. \square

In particular if \mathcal{E} is a small bicomplete elementary exact category, then there is a model category structure on \mathbf{sE} where for the set Z in Theorem 5.20 we take a generating set \mathcal{P} of tiny projective objects. We shall call this the **projective model structure on \mathbf{sE}** . We are now going to show the following

Theorem 5.21 (Model Dold-Kan for Elementary Exact Categories). *Let \mathcal{E} be a small bicomplete elementary exact category. Endow $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ and \mathbf{sE} with their projective model structures. Then the functors*

$$\Gamma : \mathbf{Ch}_{\geq 0}(\mathcal{E}) \rightarrow \mathbf{sE}$$

and

$$N : \mathbf{sE} \rightarrow \mathbf{Ch}_{\geq 0}(\mathcal{E})$$

form a Quillen equivalence.

We use the following notion:

Definition 5.22. Let \mathcal{M}, \mathcal{N} be model categories. \mathcal{M} is said to be **generated** by a collection of functors $\{F_i : \mathcal{M} \rightarrow \mathcal{N}\}_{i \in I}$ if a map $f : X \rightarrow Y$ in \mathcal{M} is a fibration (resp. weak equivalence) if and only if $F_i(f)$ is a fibration (resp. weak equivalence) for each $i \in I$.

By construction the model structure on \mathbf{sE} is generated by the functors

$$\{\mathbf{Hom}_{\mathbf{sE}}(\mathbf{s}P, -) : \mathbf{sE} \rightarrow \mathbf{sAb}\}_{P \in \mathcal{P}}$$

where we endow \mathbf{sAb} with its projective model structure.

The model structure on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ is generated by a similar set of functors:

Proposition 5.23. *Let \mathcal{E} be an elementary exact category with a projective generating set \mathcal{P} . The projective model structure on $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ is generated by the functors*

$$\{\mathbf{Hom}(S^0(P), -) : \mathbf{Ch}_{\geq 0}(\mathcal{E}) \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) : P \in \mathcal{P}\}$$

where we endow $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ with its projective model structure.

Proof. The fibrations in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$ are the degree-wise admissible epics in positive degree, and the fibrations in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ are the degree-wise epics in positive degree. Let $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$ be a morphism in $\mathbf{Ch}_{\geq 0}(\mathcal{E})$. Then the components of $\mathbf{Hom}(S^0(P), f_{\bullet})$ are $\mathrm{Hom}_{\mathcal{E}}(P, f_n)$. Now f_{\bullet} is a fibration if and only if each f_n is an admissible epimorphism for $n > 0$. This is true if and only if $\mathrm{Hom}_{\mathcal{E}}(P, f_n)$ is an epic for each $n > 0$ and each $P \in \mathcal{P}$, i.e. if and only if $\mathbf{Hom}(S^0(P), f_{\bullet})$ is a fibration for each $P \in \mathcal{P}$.

It is clear that $\mathbf{Hom}(S^0(P), \mathrm{cone}(f_{\bullet})) \cong \mathrm{cone}(\mathbf{Hom}(S^0(P), f_{\bullet}))$. Now by Corollary 3.4, $\mathrm{cone}(f_{\bullet})$ is acyclic if and only if $\mathbf{Hom}(S^0(P), \mathrm{cone}(f_{\bullet}))$ is acyclic for all $P \in \mathcal{P}$. Equivalently, f_{\bullet} is a weak equivalence if and only if $\mathbf{Hom}(S^0(P), f_{\bullet})$ is a weak equivalence for each $P \in \mathcal{P}$. \square

With these structures in hand, we will use the following result in order to prove the theorem.

Proposition 5.24. *Let $\mathcal{M}, \mathcal{N}, \mathcal{M}', \mathcal{N}'$ be model categories. Suppose \mathcal{M} is generated by functors $\{F_i : \mathcal{M} \rightarrow \mathcal{N}\}_{i \in I}$, and \mathcal{M}' is generated by functors $\{F'_i : \mathcal{M}' \rightarrow \mathcal{N}'\}_{i \in I}$. Let $G : \mathcal{M} \rightarrow \mathcal{M}'$ and $H : \mathcal{M}' \rightarrow \mathcal{M}$ be adjoint functors*

$$G \dashv H$$

Suppose also that there is a Quillen adjunction $P \dashv Q$, with $P : \mathcal{N} \rightarrow \mathcal{N}'$ and $Q : \mathcal{N}' \rightarrow \mathcal{N}$ such that for each $i \in I$ the diagram

$$\begin{array}{ccc} \mathcal{M} & \xleftarrow{H} & \mathcal{M}' \\ \downarrow F_i & & \downarrow F'_i \\ \mathcal{N} & \xleftarrow{Q} & \mathcal{N}' \end{array}$$

commutes. Then $G \dashv H$ is a Quillen adjunction.

Proof. We need to show that H preserves (acyclic) fibrations. Let f be an (acyclic) fibration in \mathcal{M}' . By assumption, for each i , $F'_i(f)$ is an (acyclic) fibration in \mathcal{N}' . Since Q is right Quillen, $Q \circ F'_i(f)$ is an (acyclic) fibration. By commutativity of the diagram

$$\begin{array}{ccc} \mathcal{M} & \xleftarrow{H} & \mathcal{M}' \\ \downarrow F_i & & \downarrow F'_i \\ \mathcal{N} & \xleftarrow{Q} & \mathcal{N}' \end{array}$$

$F_i \circ H(f)$ is an (acyclic) fibration for each $i \in I$. Again by assumption, $H(f)$ is an (acyclic) fibration. \square

Before proving the theorem, we shall make the following easy observation.

Proposition 5.25. *Let \mathcal{M} and \mathcal{M}' be model categories, and $G : \mathcal{M} \rightarrow \mathcal{M}'$ and $H : \mathcal{M}' \rightarrow \mathcal{M}$ be Quillen adjoint functors*

$$G \dashv H$$

Suppose further that

- (1) *The unit and counit maps of the adjunction are weak equivalences.*
- (2) *G preserves weak equivalences of the form $X \rightarrow HY$ where X is cofibrant and Y is fibrant.*
- (3) *H preserves weak equivalences of the form $GX \rightarrow Y$ where X is cofibrant and Y is fibrant.*

Then $G \dashv H$ is a Quillen equivalence.

Proof. Let X be a cofibrant object of \mathcal{M} and Y a fibrant object of \mathcal{M}' . Suppose that $f : GX \rightarrow Y$ is a weak equivalence. Then by assumption $HGX \rightarrow HY$ is a weak equivalence. Also by assumption $X \rightarrow HGX$ is a weak equivalence. Hence $X \rightarrow HY$ is a weak equivalence.

Conversely suppose that $X \rightarrow HY$ is a weak equivalence. Then $GX \rightarrow GHY$ is a weak equivalence by assumption. Also by assumption $GHY \rightarrow Y$ is a weak equivalence. Thus $GX \rightarrow Y$ is a weak equivalence. \square

Proof of Theorem 5.21. We first note that the following diagrams commute (up to natural isomorphism).

$$\begin{array}{ccc}
 \mathbf{sE} & \xleftarrow{\Gamma} & \mathbf{Ch}_{\geq 0}(\mathcal{E}) \\
 \downarrow \text{Hom}_{\mathbf{sE}}(\mathbf{s}P, -) & & \downarrow \text{Hom}(S^0(P), -) \\
 \mathbf{sAb} & \xleftarrow{\Gamma} & \mathbf{Ch}_{\geq 0}(\mathbf{Ab})
 \end{array}$$

$$\begin{array}{ccc}
 \mathbf{sE} & \xrightarrow{N} & \mathbf{Ch}_{\geq 0}(\mathcal{E}) \\
 \downarrow \text{Hom}_{\mathbf{sE}}(\mathbf{s}P, -) & & \downarrow \text{Hom}(S^0(P), -) \\
 \mathbf{sAb} & \xrightarrow{N} & \mathbf{Ch}_{\geq 0}(\mathbf{Ab})
 \end{array}$$

The second diagram follows from the fact that $\text{Hom}(P, -) : \mathcal{E} \rightarrow \mathbf{Ab}$ preserves kernels (and therefore intersections). The first diagram follows from the fact that $\text{Hom}(P, -) : \mathcal{E} \rightarrow \mathbf{Ab}$ preserves finite direct sums. By Proposition 5.24 the adjunction is a Quillen adjunction. Let us now check the hypotheses of Proposition 5.25. The unit and counit maps are isomorphisms. In particular they are weak equivalences. In the Dold-Kan correspondence for abelian groups, it can be shown that the functors $N : \mathbf{sAb} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ and $\Gamma : \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) \rightarrow \mathbf{sAb}$ both preserve all weak equivalences. By the commutativity of the above diagrams, this also implies that the functors $N : \mathbf{sE} \rightarrow \mathbf{Ch}_{\geq 0}(\mathcal{E})$ and $\Gamma : \mathbf{Ch}_{\geq 0}(\mathcal{E}) \rightarrow \mathbf{sE}$ also preserve all weak equivalences. \square

6. MODEL STRUCTURES ON ALGEBRAS

In this section we will let \mathcal{E} be a bicomplete, locally presentable additive category. We further assume that \mathcal{E} is endowed with an additive monoidal model structure which satisfies the monoid axiom and is combinatorial. We do not assume that either the model structure or the monoidal structure are compatible with the exact structure. An additive category satisfying all of the above assumptions will be called a **higher algebra setting (HAS)** (c.f. the notion of HAG in [TV04]). An HAS is said to be a **strong HAS** if there is a set of generating acyclic cofibrations which are split monomorphisms with trivially cofibrant cokernel.

Example 6.1. Let $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ be a locally presentable closed monoidal elementary exact category. Then $(\mathbf{Ch}_{\geq 0}(\mathcal{E}), \otimes, \underline{\text{Hom}}, S^0(k))$ is a strong HAS. If countable coproducts are admissibly coexact and countable products are admissibly exact then $(\mathbf{Ch}(\mathcal{E}), \otimes, \underline{\text{Hom}}, S^0(k))$ is also a strong HAS.

6.1. Model Structures on Monoids and Modules.

6.1.1. Modules.

Proposition 6.2. Let $(\mathcal{E}, \otimes, k)$ be a HAS and let R be a commutative monoid in \mathcal{E} . Then with its transferred model structure and induced closed symmetric monoidal structure, $(R - \mathbf{Mod}, \otimes_R, \underline{\text{Hom}}_R)$ is a HAS. If $(\mathcal{E}, \otimes, k)$ is a strong HAS then so is $(R - \mathbf{Mod}, \otimes_R, \underline{\text{Hom}}_R)$.

Proof. The transferred model structure exists by Theorem B.17. and is cofibrantly generated by Corollary B.16. Also by Theorem B.17 it is monoidal and satisfies the monoid axiom. $R - \mathbf{Mod}$ is locally presentable by [Mes14]. Finally, let I be a set of generating acyclic cofibrations for \mathcal{E} which are split monomorphisms. Then $\{\text{id}_R \otimes i : i \in I\}$ is a set of generating acyclic cofibrations in $R - \mathbf{Mod}$. Tensoring with R clearly preserves split exactness of a sequence, so $R - \mathbf{Mod}$ also has a set of generating acyclic cofibrations which are split monomorphisms with trivially cofibrant cokernel. \square

Let $(\mathcal{E}, \otimes, \underline{\text{Hom}}, k)$ be a locally presentable closed monoidal elementary exact category. Suppose that countable coproducts are admissibly coexist and countable products are admissibly exact and let R be a commutative monoid in \mathcal{E} . By Proposition 3.23 $R\text{-}\mathbf{Mod}$ is again an elementary exact category, and it is locally presentable.. Thus $\mathbf{Ch}(R\text{-}\mathbf{Mod})$ is equipped with a combinatorial projective model structure, and has a set of generating acyclic cofibrations which are split monomorphisms with trivially cofibrant cokernel. However the induced monoidal structure on $R\text{-}\mathbf{Mod}$, namely $-\otimes_R -$ need not be compatible with the exact structure. This one of our motivations for considering higher algebra settings rather than just pseudo-compatible model structures on monoidal exact categories. We also want to consider model structures on modules over commutative differential graded algebras.. It is however useful to know that in these cases the model structures are left pseudo-compatible (resp. compatible).

6.1.2. *Associative Monoids.* The next result follows immediately from Theorem B.17

Proposition 6.3. *Let \mathcal{E} be a HAS. Then the transferred model structure exists on $\mathbf{Ass}(\mathcal{E})$.*

6.1.3. *Commutative Monoids.* Now we turn to commutative monoids. If a (strong) HAS \mathcal{E} is enriched over $\mathbf{Vect}_{\mathbb{Q}}$ rather than just \mathbf{Ab} , we shall call it a (strong) \mathbb{Q} -HAS.

Proposition 6.4. *Let \mathcal{E} be a strong \mathbb{Q} -HAS. Then the transferred model structure on $\mathbf{Comm}(\mathcal{E})$ exists.*

Proof. The forgetful functor $\mathbf{Comm}(\mathcal{E}) \rightarrow \mathcal{E}$ preserves filtered colimits. Thus we may apply Corollary B.16. The transferred model structure exists on $\mathbf{Ass}(\mathcal{E})$ by Proposition 6.3. In particular the functor $T : \mathcal{E} \rightarrow \mathbf{Ass}(\mathcal{E})$ preserves acyclic cofibrations. By Proposition A.1, for any map $X \rightarrow Y$ in \mathcal{E} , the map $S(X) \rightarrow S(Y)$ is a retract of $T(X) \rightarrow T(Y)$. In particular if $X \rightarrow Y$ is an acyclic cofibration in \mathcal{E} , then $S(X) \rightarrow S(Y)$ is a weak equivalence in \mathcal{E} . Now suppose g is a generating acyclic cofibration in \mathcal{E} . We may assume g is an inclusion as a direct summand, i.e. of the form $X \rightarrow X \oplus Z$ where Z is trivially cofibrant. Since S is a left adjoint it preserves colimits, so $S(X \oplus Z) \cong S(X) \otimes S(Z)$, and $S(g)$ is the map $\text{id}_{S(X)} \otimes 1_{S(Z)}$ where $1_{S(Z)}$ is the unit of the commutative monoid $S(Z)$. Consider a push-out diagram

$$\begin{array}{ccc} S(X) & \longrightarrow & A \\ \downarrow S(g) & & \downarrow S(g)' \\ S(X) \otimes S(Z) & \longrightarrow & B \end{array}$$

Then B is isomorphic to $A \otimes_{S(X)} (S(X) \otimes S(Z)) \cong A \otimes S(Z)$ and under this isomorphism $S(g)'$ is $\text{id}_A \otimes 1_{S(Z)}$. Any transfinite composition of such maps will again be of the form $t : A \rightarrow A \otimes S(Y)$ with Y trivially cofibrant, since both \otimes and S preserve colimits and coproducts of trivially cofibrant objects are trivially cofibrant. Now $0 \rightarrow Y$ is a weak equivalence between cofibrant objects. Thus $k = S(0) \rightarrow S(Y)$ is a weak equivalence. Moreover tensor products, coproducts, and retracts of (trivially) cofibrant objects are (trivially) cofibrant. Hence $S(Y)$ is trivially cofibrant. Since the model structure on \mathcal{E} is monoidal, $(-) \otimes A$ preserves weak equivalences between cofibrant objects. In particular t is a weak equivalence. \square

6.1.4. *Lie Monoids.* Finally we turn to Lie monoids.

Proposition 6.5. *Let $(\mathcal{E}, \otimes, k)$ be a \mathbb{Q} -HAS. Then the transferred model structure exists on $\mathbf{Lie}(\mathcal{E})$*

Proof. Let $f : X \rightarrow Y$ be a generating trivial cofibration in \mathcal{E} and suppose

$$\begin{array}{ccc} L(X) & \xrightarrow{f} & A \\ \downarrow L(g) & & \downarrow g' \\ L(Y) & \xrightarrow{f'} & B \end{array}$$

is a pushout diagram in $\mathbf{Lie}(\mathcal{E})$. Since U is a left-adjoint the following diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{U(f)} & U(A) \\ \downarrow T(g) & & \downarrow U(g') \\ T(Y) & \xrightarrow{U(f')} & U(B) \end{array}$$

is a pushout in $\mathbf{Ass}(\mathcal{E})$. Now as a left adjoint, the functor U preserves colimits. Thus if m is a transfinite composition of pushouts of images $L(g)$ of generating acyclic cofibrations g , then $U(m)$ is a transfinite composition of pushouts of images $T(g)$ of generating acyclic cofibrations g . By Theorem 6.3 and Theorem B.14 $U(m)$ is acyclic. But m is a retract of $U(m)$ by Theorem A.2. Hence m is also a weak equivalence. \square

7. CONCLUDING REMARKS

7.1. Relaxing the Assumptions. We are being somewhat lazy by assuming the existence of abelian embeddings. Many of the results we prove using this assumption likely also hold under other, sometimes easier to check assumptions such as existence of kernels, idempotent completeness or even just weak idempotent completeness. Buehler makes a similar remark with regard to his result in [Büh11].

7.2. Comparisons With Other Work.

7.2.1. The Injective Model Structure.

Definition 7.1. Let \mathcal{E} be an exact category. If it exists, the *injective model structure* on $\mathbf{Ch}_*(\mathcal{E})$, for $*$ $\in \{+, b, \emptyset\}$ is the model structure in which

- Weak equivalences are quasi-isomorphisms.
- Cofibrations are degree-wise admissible monics.
- Fibrations are maps which have the right lifting property with respect to acyclic cofibrations.

In [Št'12] introduces the notions of efficient exact categories and exact categories of Grothendieck type.

Definition 7.2. A exact category \mathcal{E} is said to be *efficient* if

- (1) \mathcal{E} is weakly idempotent complete.
- (2) Transfinite compositions of admissible monics exist and are admissible monics.
- (3) Every object of \mathcal{E} is tiny relative to the class of all admissible monics.
- (4) \mathcal{E} has an admissible generator.

Remark 7.3. Locally presentable elementary exact categories are efficient.

Definition 7.4. Let \mathcal{E} be an exact category, and \mathcal{S} a class of objects in \mathcal{E} . An *\mathcal{S} -filtration* is diagram $X : \lambda \rightarrow \mathcal{E}$ for some ordinal λ , such that $X_0 = 0$ and for each $\alpha + 1 < \lambda$, there is a short exact sequence

$$0 \longrightarrow X_\alpha \xrightarrow{f_{\alpha, \alpha+1}} X_{\alpha+1} \longrightarrow S_\alpha \longrightarrow 0$$

with $S_\alpha \in \mathcal{S}$. An object $X \in \mathcal{E}$ is said to be *\mathcal{S} -filtered* if $0 \rightarrow X$ is the transfinite composition of some \mathcal{S} -filtration. The class of all \mathcal{S} -filtered objects is denoted by $\mathbf{Filt}\mathcal{S}$. A class \mathcal{F} of objects in \mathcal{E} is said to be *deconstructible* if there exists a set \mathcal{S} of objects of \mathcal{E} such that $\mathcal{F} = \mathbf{Filt}\mathcal{S}$.

Definition 7.5. An exact category \mathcal{E} is said to be *of Grothendieck type* if it is efficient and is deconstructible in itself.

Theorem 7.11 in [Št'12] then says the following.

Theorem 7.6. *Let \mathcal{E} be an exact category of Grothendieck type such that \mathfrak{W} , the class of acyclic complexes in $\mathbf{Ch}(\mathcal{E})$ is deconstructible in $\mathbf{Ch}(\mathcal{E})$. Then the injective model structure exists on $\mathbf{Ch}(\mathcal{E})$. Moreover it is a compatible model structure.*

It is not clear to us whether the categories we are interested in, namely \mathbf{CBorn}_k and $\mathbf{Ind}(\mathbf{Ban}_k)$ are of Grothendieck type.

APPENDIX A. ALGEBRA IN SYMMETRIC MONOIDAL CATEGORIES

Throughout this section $(\mathcal{C}, \otimes, k)$ is a symmetric monoidal category, with monoidal functor \otimes . The symmetric braiding will be denoted by σ . We further assume that \mathcal{C} is finitely bicomplete. What follows is largely standard. Much of it can be found in [BBK13] for example.

A.1. Associative Monoids. We denote the category of (unital) associative monoids internal to \mathcal{C} by $\mathbf{Ass}(\mathcal{C})$. There is a faithful forgetful functor $|-|_{\mathbf{Ass}} : \mathbf{Ass}(\mathcal{C}) \rightarrow \mathcal{C}$. If \mathcal{C} has countable products then $|-|$ has a left adjoint T which can be constructed explicitly. Namely for $V \in \mathcal{C}$, set

$$T_n(V) = V^{\otimes n}$$

$$T(V) = \bigoplus_{n=0}^{\infty} T_n(V)$$

where by definition $T_0(V) = V^{\otimes 0} = k$. Now \otimes preserves colimits in each variable, so

$$T(V) \otimes T(V) \cong \bigoplus_{m,n=0}^{\infty} T_m(V) \otimes T_n(V)$$

The multiplication

$$m : T(V) \otimes T(V) \rightarrow T(V)$$

is defined on the summand $T_m(V) \otimes T_n(V)$ by the composition

$$T_m(V) \otimes T_n(V) \cong T_{m+n}(V) \rightarrow T(V)$$

where the isomorphism $T_m(V) \otimes T_n(V) = V^{\otimes m} \otimes V^{\otimes n} \cong V^{\otimes(m+n)} = T_{m+n}(V)$ is the natural isomorphism. The identity is given by the inclusion $e : k = T_0(V) \rightarrow T(V)$. m and e endow $T(V)$ with the structure of a unital associative monoid. It is clear that $V \rightarrow T(V)$ is functorial in V , and it is straightforward to check that T is left adjoint to $|-|$.

A.2. Commutative Monoids. We denote the category of (unital) commutative monoids by $\mathbf{Comm}(\mathcal{C})$. If \mathcal{C} has finite coequalizers and countable coproducts then the forgetful functor $|-|_{\mathbf{Comm}} : \mathbf{Comm}(\mathcal{C}) \rightarrow \mathcal{C}$ has a left-adjoint, which can be constructed explicitly as follows. The symmetric group on n letters Σ_n acts on $T_n(V) = V^{\otimes n}$. Let $S_n(V) = T_n(V)_{\Sigma_n}$ be the coinvariants for this action. We then set

$$S(V) = \bigoplus_{n=0}^{\infty} S_n(V)$$

The associative monoid structure on $T(V)$ descends to an associative monoid structure on $S(V)$. One checks easily that it is commutative and that it is a left adjoint.

A.3. Modules. Given objects A and B of $\mathbf{Ass}(\mathcal{C})$ we denote by $A - \mathbf{Mod}$ the category of left modules for A , by $\mathbf{Mod} - A$ the category of right modules for A , and by $A - \mathbf{Mod} - B$ the category of $A - B$ bimodules.

There is a forgetful functor $|-|_{A - \mathbf{Mod}} : A - \mathbf{Mod} \rightarrow \mathcal{E}$. This functor has a left adjoint. It sends an object E to the object $A \otimes E$ with the obvious left action of A .

Let E be a right A -module with action morphism

$$a_E : E \otimes A \rightarrow E$$

and F a left A -module with action morphism

$$a_F : A \otimes F \rightarrow F$$

If the category \mathcal{C} has finite equalisers, then we define

$$E \otimes_A F$$

to be the equaliser of the maps

$$\begin{array}{ccc} & \xrightarrow{\alpha_E} & \\ E \otimes A \otimes F & & E \otimes F \\ & \xleftarrow{\alpha_F} & \end{array}$$

This defines a bifunctor

$$\otimes_A : \mathbf{Mod} - A \times A - \mathbf{Mod} \rightarrow \mathcal{C}$$

If E is a $B - A$ bimodule and F is an $A - C$ bimodule, then $E \otimes_A F$ is naturally a $B - C$ -bimodule, i.e. \otimes_A gives a bifunctor

$$B - \mathbf{Mod} - A \times A - \mathbf{Mod} - C \rightarrow B - \mathbf{Mod} - C$$

If A is a commutative monoid then this gives a bifunctor

$$A - \mathbf{Mod} \times A - \mathbf{Mod} \rightarrow A - \mathbf{Mod}$$

which endows $A - \mathbf{Mod}$ with a monoidal structure.

Suppose further that the monoidal structure is closed, and let $\underline{\mathbf{Hom}}(-, -)$ denote the internal hom functor. Then one can also construct an internal hom, $\underline{\mathbf{Hom}}_A(-, -)$ functor on $A - \mathbf{Mod}$ by a similar method as used to construct \otimes_A . This makes $(\mathcal{C}, \otimes_A, \underline{\mathbf{Hom}}_A(-, -), A)$ a closed monoidal category. See for example [BBK13] for details.

A.4. Lie Monoids. Now we suppose $(\mathcal{E}, \otimes, k)$ is a monoidal additive category. Then one can define the category of Lie monoids internal to \mathcal{E} . Denote the symmetric braiding by σ . A **Lie monoid** in \mathcal{E} is a pair $(L, [-, -])$ consisting an object L of \mathcal{E} together with a morphism $[-, -] : L \otimes L \rightarrow L$ satisfying the Jacobi identity

$$[-, [-, -]] + [-, [-, -]] \circ (\mathrm{id}_L \otimes \sigma_{L,L}) + [-, [-, -]] \circ (\sigma_{L,L} \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \sigma_{L,L}) = 0$$

and the antisymmetry condition

$$[-, -] + [-, -] \otimes \sigma_{L,L} = 0$$

Morphisms of Lie monoids are defined in the obvious way. This gives a category $\mathbf{Lie}(\mathcal{E})$ of Lie monoids internal to \mathcal{C} .

There is of course a forgetful functor $|-|_{\mathbf{Lie}} : \mathbf{Lie}(\mathcal{E}) \rightarrow \mathcal{E}$. If \mathcal{E} is enriched over $\mathbf{Vect}_{\mathbb{Q}}$ rather than \mathbf{Ab} we will also see that this functor has a left adjoint L which can be constructed explicitly.

Now let A be an associative monoid in \mathcal{E} with multiplication m . Define $[-, -] : A \otimes A \rightarrow A$ by $[-, -] = m - m \circ \sigma_{A,A}$. It is easy to see that $(A, [-, -])$ is a Lie monoid. Moreover this structure is clearly functorial, and we get a faithful functor $\mathbf{Ass}(\mathcal{E}) \rightarrow \mathbf{Lie}(\mathcal{E})$. As we shall see later, if \mathcal{E} is enriched over $\mathbf{Vect}_{\mathbb{Q}}$ then this functor has a left adjoint U .

A.5. Algebra in $\mathbf{Vect}_{\mathbb{Q}}$ -Enriched Symmetric Monoidal Categories. We now assume that our monoidal additive category \mathcal{E} is enriched over $\mathbf{Vect}_{\mathbb{Q}}$ rather than just \mathbf{Ab} . We also assume that \mathcal{E} is finitely bicomplete and has countable coproducts. Let us relate the functors U, L, T, S .

The easiest identity is $U \circ L \cong T$. This follows from the fact that both $U \circ L$ and T are left adjoints to the forgetful functor $|-|_{\mathbf{Ass}} : \mathbf{Ass}(\mathcal{E}) \rightarrow \mathcal{E}$.

Now consider T and S . The following is an easy generalisation of the same fact for \mathbb{Q} -vector spaces. It is done for dg -vector spaces in [Qui69] for example.

Proposition A.1. *The natural transformation $|-|_{\mathbf{Ass}} \circ T \rightarrow |-|_{\mathbf{Comm}} \circ S$ admits a section.*

Proof. Let V be an object of \mathcal{E} . Define a map $\rho_V : T(V) \rightarrow T(V)$ of graded objects in \mathcal{E} by

$$\rho_{V, n} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma : T_n(V) \rightarrow T_n(V)$$

This clearly induces a map

$$T_n(V)_{\Sigma_n} = S_n(V) \rightarrow T_n(V)$$

which is a section of the projection $T_n(V) \rightarrow S_n(V)$. It is also clear that ρ_V is natural in V , i.e. we get a natural transformation $\rho : |-|_{\mathbf{Comm}} \circ S \rightarrow |-|_{\mathbf{Ass}} \circ T$ which is a section of $|-|_{\mathbf{Ass}} \circ T \rightarrow |-|_{\mathbf{Comm}} \circ S$ \square

Let us now explain how U and S are related. In [DEF⁺99] it is shown that if \mathcal{E} is \mathbb{Q} -linear then a left adjoint U to the forgetful functor $\mathbf{Ass}(\mathcal{E}) \rightarrow \mathbf{Lie}(\mathcal{E})$ exists, and there is a natural isomorphism

$$|-|_{\mathbf{Ass}} \circ U \cong |-|_{\mathbf{Comm}} \circ S \circ |-|_{\mathbf{Lie}}$$

$U(L)$ is called the **universal enveloping algebra** of L . The proof in fact works in the following setup

Theorem A.2 (Poincaré-Birkhoff-Witt). *Let $(\mathcal{E}, \otimes, k)$ a monoidal additive category enriched over $\mathbf{Vect}_{\mathbb{Q}}$ with countable coproducts and finite coequalizers. Then a left adjoint U to the forgetful functor $\mathbf{Ass}(\mathcal{C}) \rightarrow \mathbf{Lie}(\mathcal{C})$ exists, and there is a natural isomorphism*

$$|-|_{\mathbf{Ass}} \circ U \cong |-|_{\mathbf{Comm}} \circ S \circ |-|_{\mathbf{Lie}}$$

Corollary A.3. *Let \mathfrak{g} be a Lie monoid and let $i : \mathfrak{g} \rightarrow U(\mathfrak{g})$ denote the natural map in \mathcal{E} . Then the map $\mathfrak{g} \rightarrow \text{Im}(i)$ is an isomorphism.*

Finally we relate U and L . First we give an explicit construction of L . Consider the tensor algebra $T(V)$ as a Lie algebra with Lie bracket $[-, -]$ the one induced from the associative algebra structure. Let $L_0(V) = V \hookrightarrow T(V)$. Inductively define a subobject $L_{r+1}(V)$ of $T(V)$ as the image of the restriction of $[-, -]$ to $V \otimes L_r(V)$. Define

$$L(V) = \bigoplus_{r=0}^{\infty} L_r(V)$$

The Lie bracket on $T(V)$ pulls back to one on $L(V)$. The construction is clearly functorial. To see that it is a left adjoint we follow the method of [S⁺71]. Suppose \mathfrak{g} is a Lie monoid and $V \rightarrow \mathfrak{g}$ is a morphism in \mathcal{E} . This induces a morphism $V \rightarrow \mathfrak{g} \rightarrow U(\mathfrak{g})$ and therefore a morphism of associative algebras $T(V) \rightarrow U(\mathfrak{g})$. The image of $L(V)$ under this map is clearly contained in the image of \mathfrak{g} in $U(\mathfrak{g})$. But by Corollary A.3 this is isomorphic to \mathfrak{g} . Thus we get a lift of $V \rightarrow \mathfrak{g}$ to a map of Lie algebras $L(V) \rightarrow \mathfrak{g}$. Such a map is clearly unique.

We are going to show that the natural inclusion $L(V) \hookrightarrow T(V)$ is split. First we introduce some notation. Let \mathfrak{g} be a Lie monoid with bracket $[-, -]$. Define $[-, -]_n : \mathfrak{g}^{\otimes n} \rightarrow \mathfrak{g}$ inductively as follows. We set $[-, -]_1 = [-, -]$ and define $[-, -]_{n+1}$ to be the composite.

$$\mathfrak{g}^{\otimes n+1} \xrightarrow{id_{\mathfrak{g}} \otimes [-, -]_n} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{[-, -]} \mathfrak{g}$$

We then get the following result, which is a generalisation of Lemma 2.2 in [Qui69].

Lemma A.4. *The graded natural transformation $\rho : |-|_{\mathbf{Ass}} \circ T \rightarrow |-|_{\mathbf{Lie}} \circ L$ of graded objects in \mathcal{E} given by*

$$\rho_n = \begin{cases} 0 & n = 0 \\ [-, -]_n & n > 0 \end{cases}$$

is a left inverse for the map $L(V) \rightarrow T(V)$.

Proof. Fix an object V of \mathcal{E} . Denote the Lie bracket on $L(V)$ by $[-, -]$. Define a Lie monoid endomorphism D of $L(V)$ whose action on $L_n(V)$ is multiplication by n . Consider the Lie monoid $L'(V) = L(V) \oplus k$ with Lie bracket

$$[-, -]': (L(V) \oplus k) \otimes (L(V) \oplus k) \cong L(V) \otimes L(V) \oplus L(V) \oplus L(V) \oplus k \rightarrow L(V) \oplus k$$

given by the matrix

$$\begin{pmatrix} [-, -] & D & -D & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The inclusion $L(V) \rightarrow L(V) \oplus k$ is a morphism of Lie monoids whose image is an ideal of $L(V) \oplus k$. Thus $L(V) \oplus k$ is an $L(V)$ -module, and hence a $U(L(V)) = T(V)$ module. Consider the composition

$$T(V) \cong T(V) \otimes k \rightarrow T(V) \otimes L'(V) \rightarrow L'(V)$$

In degree n this map is $n\rho_n$. But when restricted to $L_n(V)$ it is given by $D_n = \text{id}_{L_n(V)}$. Thus $\rho_n|_{L_n(V)} = \text{id}_{L_n(V)}$. \square

APPENDIX B. MODEL CATEGORIES

B.1. Weak Factorization Systems and Model Structures. Here we briefly recall the definition of a model structure by means of weak factorisation systems. Details can be found in [Rie14].

Definition B.1. Let \mathcal{C} be a class of morphisms in a category \mathcal{M} . A morphism f in \mathcal{M} is said to have the **left lifting property** with respect to \mathcal{C} if in any diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow f & & \downarrow c \\ B & \longrightarrow & D \end{array}$$

with $c \in \mathcal{C}$, there exists a morphism $h : B \rightarrow C$ such that the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow f & \nearrow h & \downarrow c \\ B & \longrightarrow & D \end{array}$$

We denote the class of all morphisms which have the left-lifting property with respect to \mathcal{C} by \mathcal{C}^\perp .

Dually one defines the morphisms having the right lifting property with respect to \mathcal{C} . The class of all such morphisms is denoted \mathcal{C}^\nearrow .

The following is straightforward

Proposition B.2. Let \mathcal{C} be a class of morphisms in a category \mathcal{M} . Then \mathcal{C}^\perp is closed under retracts, push-outs and transfinite composition (whenever they exist).

Proof. See [Rie14] Lemma 11.1.4. \square

Definition B.3. A **weak factorisation system** on a category \mathcal{C} is a pair $(\mathcal{L}, \mathcal{R})$ such that

(1) Any map in \mathcal{C} can be factored as a map in \mathcal{L} followed by a map in \mathcal{R} .

(2) $\mathcal{L} = \mathcal{R}^\perp$ and $\mathcal{R} = \mathcal{L}^\nearrow$.

A weak factorisation system is said to be **functorial** if the factorisation in (1) can be made functorial.

We can now give a definition of the notion of a model structure in terms of weak factorisation systems.

Definition B.4. A **model structure** on a category \mathcal{M} is a collection of three wide subcategories $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ such that

(1) The class \mathcal{W} satisfies the 2-out-of-6 property (see [Rie14]).

(2) Both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorization systems.

We do not assume completeness or cocompleteness of \mathcal{M} .

Definition B.5. A model structure on a category \mathcal{M} is said to be **functorial** if the factorisation systems are functorial.

Definition B.6. A **(functorial) model category** a category together with a (functorial) model structure.

B.2. Cofibrant Generation. We state here our conventions regarding cofibration generation.

Definition B.7. Let \mathcal{C} be a category. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ on \mathcal{C} is said to be **cofibrantly small** if there is a set I of maps in \mathcal{L} such that $\mathcal{R} = I^{\mathcal{R}}$. I is called a set of **generating morphisms**. If in addition I admits the small object argument then the weak factorisation system is said to be **cofibrantly generated**. If I can be chosen such that the domains are tiny, then the weak factorisation system is said to be **cellular**. If \mathcal{C} is locally presentable and cofibrantly small, then the weak factorisation system is said to be **combinatorial**. A model category $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is said to be cofibrantly small cofibrantly generated/ cellular/ combinatorial if both the weak factorisation systems $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are cofibrantly small/ cellular/ combinatorial.

Remark B.8. A cofibrantly small weak factorisation system (resp. model structure) on a locally presentable category is automatically cofibrantly generated.

B.3. Monoidal Model Categories.

Definition B.9. Let $\mathcal{M}, \mathcal{N}, \mathcal{P}$ be model categories. A bifunctor $- \otimes - : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$ is said to be **left Quillen** if whenever $i : m \rightarrow m'$ and $j : n \rightarrow n'$ are cofibrations then so is $i \hat{\otimes} j$, and it is an acyclic cofibration if either i or j is. Here $i \hat{\otimes} j$ is the following map

$$\begin{array}{ccc}
 m \otimes n & \xrightarrow{i \otimes 1} & m' \otimes n \\
 \downarrow 1 \otimes j & & \downarrow 1 \otimes j \\
 m \otimes n' & \xrightarrow{\quad} & P \\
 & \searrow i \hat{\otimes} j & \downarrow \\
 & & m' \otimes n'
 \end{array}$$

(Note: The diagram shows a square with a diagonal arrow from $m \otimes n'$ to $m' \otimes n'$ labeled $i \otimes 1$, and a curved arrow from $m' \otimes n$ to $m' \otimes n'$ labeled $1 \otimes j$. The central square is a pushout.)

where the square is a push out.

Definition B.10. A **(closed) monoidal model category** is a (closed) symmetric monoidal category $(\mathcal{V}, \otimes, k)$ $((\mathcal{V}, \otimes, k, \underline{Hom}))$ with a model structure so that the monoidal product is a left Quillen bifunctor, and the maps

$$Q(k) \otimes v \rightarrow k \otimes v \cong v$$

and

$$v \otimes Q(k) \rightarrow v \otimes k \cong v$$

are weak equivalences whenever v is cofibrant. Here Q is the cofibrant replacement functor.

Another condition that is often asked of a monoidal model category is that it satisfies the so-called monoid axiom. Under certain additional technical assumptions on the model category, this guarantees the existence of a model structure on the category of algebras over any cofibrant operad.

Definition B.11. A monoidal model category $(\mathcal{V}, \otimes, k, \underline{Hom})$ is said to satisfy the **monoid axiom** if every morphisms which is obtained a a transfinite composition of pushouts of tensor products of acyclic cofibrations with any object is a weak equivalence.

B.4. Transferred Model Structures.

Definition B.12. Let \mathcal{D} and \mathcal{E} be categories with \mathcal{D} a model category. Suppose $F : \mathcal{D} \rightarrow \mathcal{E}$ and $G : \mathcal{E} \rightarrow \mathcal{D}$ are functors with $F \dashv G$. If it exists, the **transferred model structure** on \mathcal{E} is the one defined as follows.

- (1) A map f in \mathcal{E} is a weak equivalence precisely if $G(f)$ is a weak equivalence in \mathcal{D} .
- (2) A map f in \mathcal{E} is a fibration precisely if $G(f)$ is a fibration in \mathcal{D} .
- (3) A map f in \mathcal{E} is a cofibration precisely if it has the left lifting property with respect to acyclic cofibrations.

Remark B.13. *If the transferred model structure exists on \mathcal{E} then $F \dashv G$ is a Quillen adjunction.*

We need the following important result, which is Theorem 3.3 in [Cra95].

Theorem B.14. *Let \mathcal{D} and \mathcal{E} be categories, with \mathcal{D} a cocomplete cellular model category and \mathcal{E} having finite limits and all colimits. Suppose $F : \mathcal{D} \rightarrow \mathcal{E}$ and $G : \mathcal{E} \rightarrow \mathcal{D}$ are functors with $F \dashv G$. If F preserves tiny objects, then the transferred model structure on \mathcal{E} exists if and only if the weak equivalences in \mathcal{E} contain any sequential colimit of pushouts of images $F(g)$, where g is allowed to vary over the generating trivial cofibrations in \mathcal{D} . Moreover the transferred model structure is cellular.*

Remark B.15. *Note that in [Cra95] it is actually proved that if an adjunction satisfying the above condition then the transferred model structure exists. The converse is clear however since as a left Quillen functor F preserves acyclic cofibrations and colimits.*

We will actually use the following immediate corollary.

Corollary B.16. *Let \mathcal{D} and \mathcal{E} be categories, with \mathcal{D} a cocomplete cellular model category and \mathcal{E} having finite limits and all colimits. Suppose $F : \mathcal{D} \rightarrow \mathcal{E}$ and $G : \mathcal{E} \rightarrow \mathcal{D}$ are functors with $F \dashv G$. If G preserves filtered colimits, then the transferred model structure on \mathcal{E} exists if and only if the weak equivalences in \mathcal{E} contain any transfinite composition of pushouts of images $F(g)$, where g is a generating trivial cofibration in \mathcal{D} . Moreover the transferred model structure is cellular.*

B.5. Algebra in Monoidal Model Categories. Let (C, \otimes) be a monoidal model category. We recall here a major result regarding the existence of transferred model structures on categories of monoids and modules internal to C .

Theorem B.17 ([SS00]). *Let (C, \otimes) be a bicomplete monoidal model category and R a monoid object in C . Suppose that*

- (1) *(C, \otimes) satisfies the monoid axiom.*
- (2) *C is a combinatorial model category.*

Then

- (1) *The transferred model structure on $R - \mathbf{Mod}$ exists and is cofibrantly generated.*
- (2) *If R is commutative, then the transferred model structure on $R - \mathbf{Mod}$ is monoidal and satisfies the monoid axiom.*
- (3) *If R is commutative then the transferred model structure exists on the category of monoids in $R - \mathbf{Mod}$. Moreover it is cofibrantly generated. Every cofibration of R -algebras whose source is cofibrant is also a cofibration of R -modules.*

Proof. This is Theorem 4.1 in [SS00]. □

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